Many body localization

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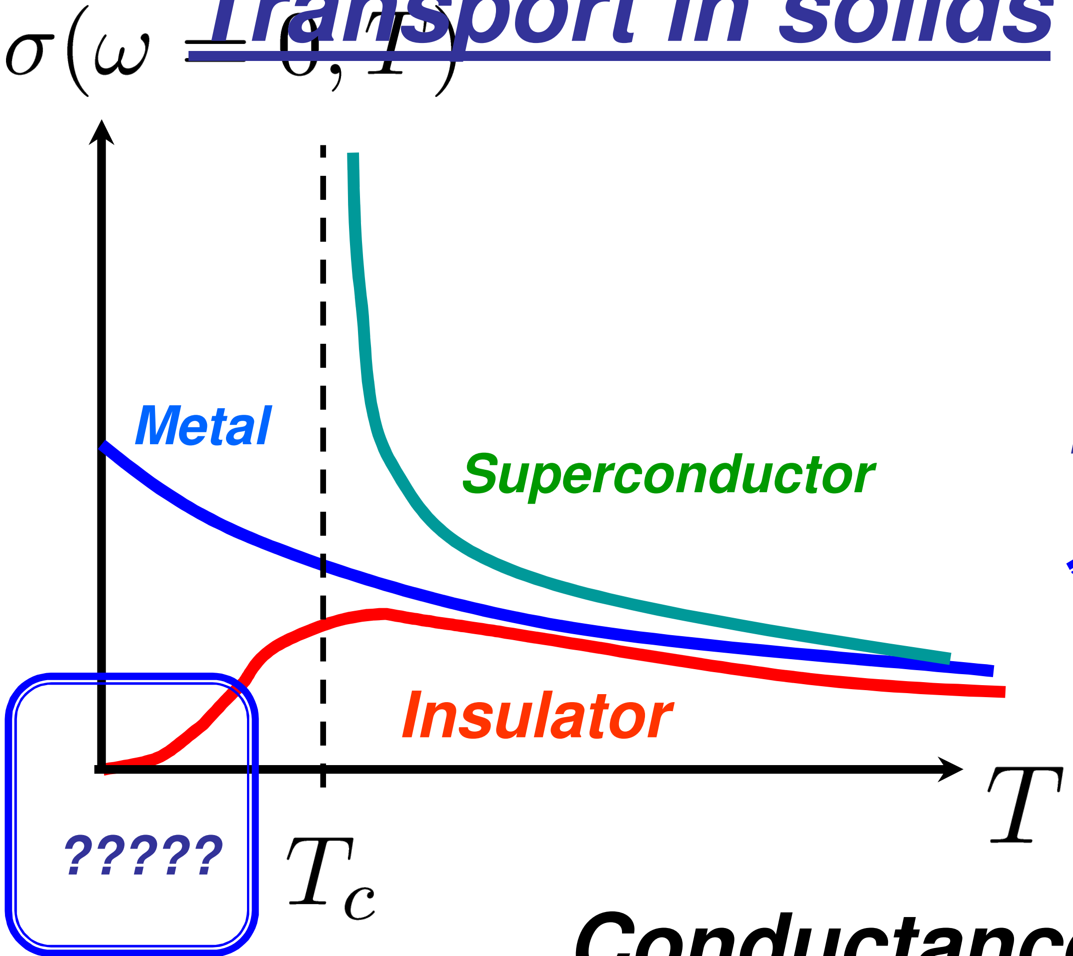
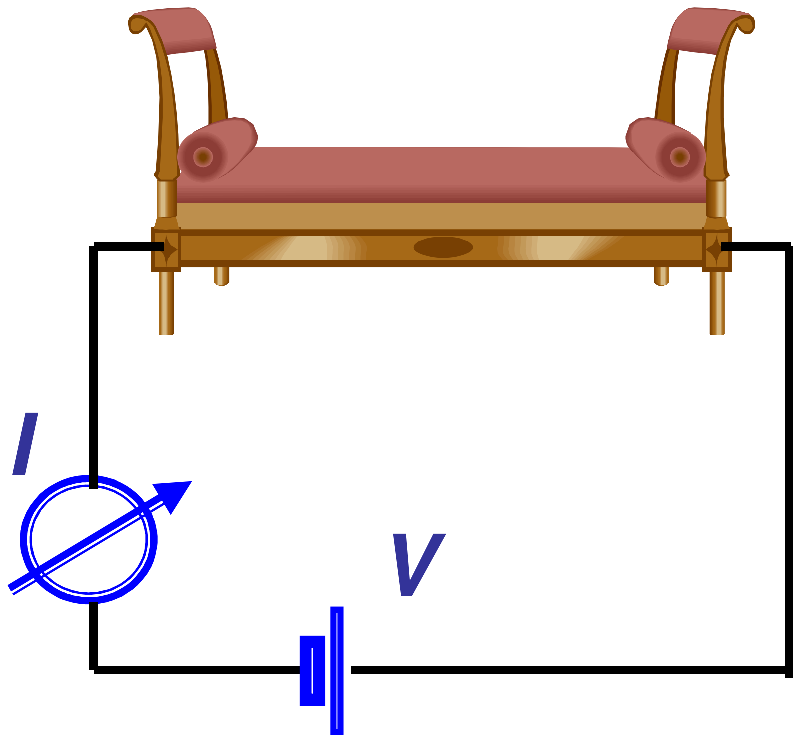
Detailed paper (fermions): Annals of Physics 321 (2006) 1126-1205
Shorter version: cond-mat/0602510; chapter in "Problems of CMP"

Bosons: NATURE PHYSICS 6 (2010) 900-904

Lewiner Institute of Theoretical Physics, Colloquium, December 23rd, 2010

Transport in solids





Conductance: $G(\omega, T) = \left.\frac{I}{V}\right|_{V\to 0}$

Conductivity: $G(\omega, T) = \sigma(\omega, T)\frac{L_x L_y}{L_z}$

Outline:

- Formulation and history of the problem
- Results for fermionic system
- Effective model
- Technique
- Stability of the metal
- Stability of the many-body insulator

- Metal insulator transition
- Extension for weakly interacting bosons in 1D.

<u>Problem:</u> can ***e-e*** interaction **<u>alone</u>** sustain finite conductivity in a localized system?

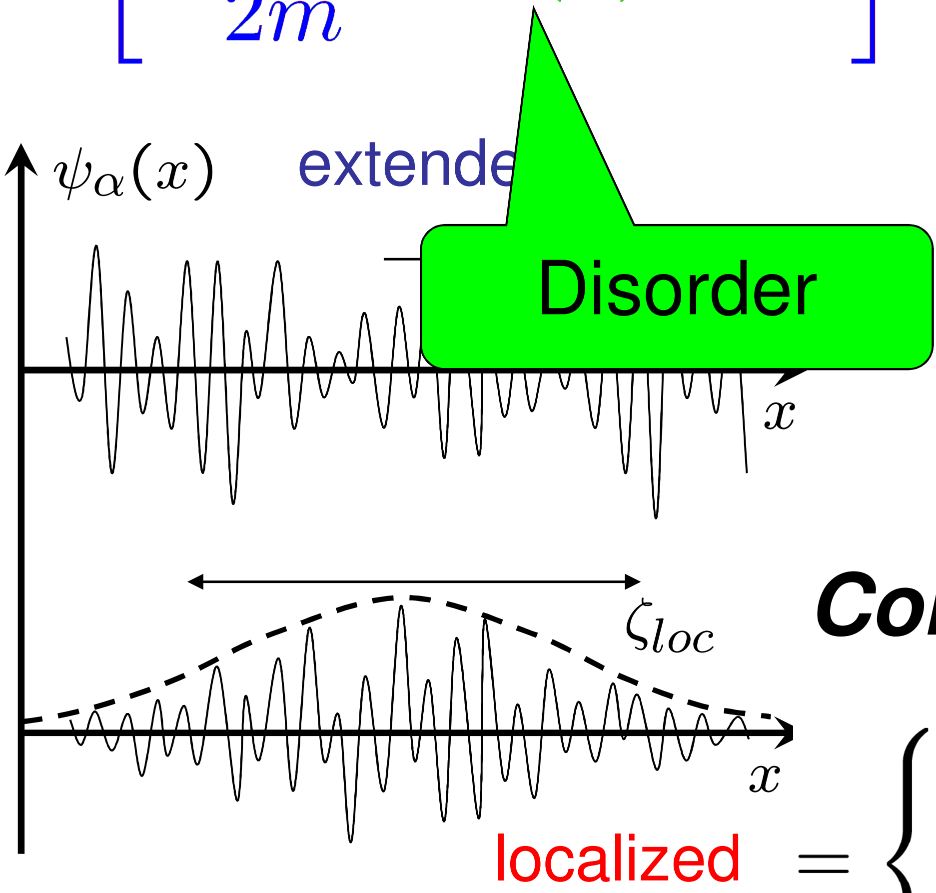
<u>Given:</u>

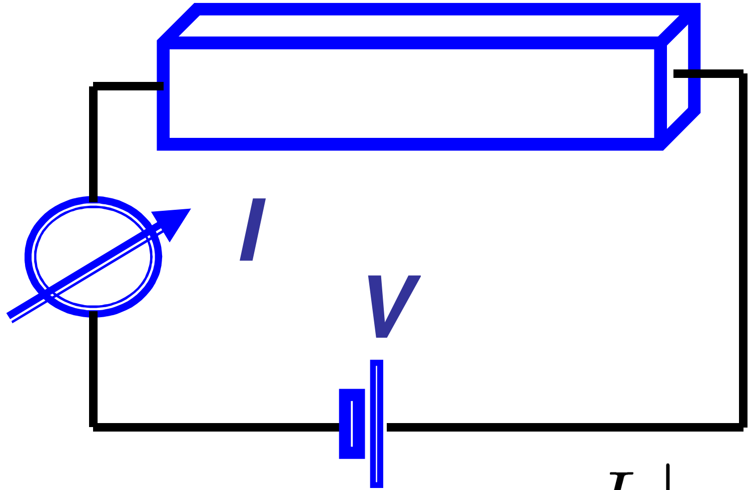
1. All one-electron states are localized
2. Electrons interact with each other
3. The system is closed (no phonons)
4. Temperature is low but finite

<u>Find:</u> DC conductivity $\sigma(T,\omega=0)$ (zero or finite?)

1. Localization of single-electron wave-functions:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m}+U(\boldsymbol{r})-\epsilon_F\right]\psi_\alpha(\boldsymbol{r})=\xi_\alpha\psi_\alpha(\boldsymbol{r})$$





Conductance

$$G=\left.\frac{I}{V}\right|_{V\to 0}=\begin{cases}\sigma\,\dfrac{L_xL_y}{L_z}\,; & \text{extended}\\ \propto\exp\left(-L_z/\zeta_{loc}\right); & \text{localized}\end{cases}$$

Most of the knowledge is based on extensions and Improvements of:

'SICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1

Absence of Diffusion in Certain Random Lattices

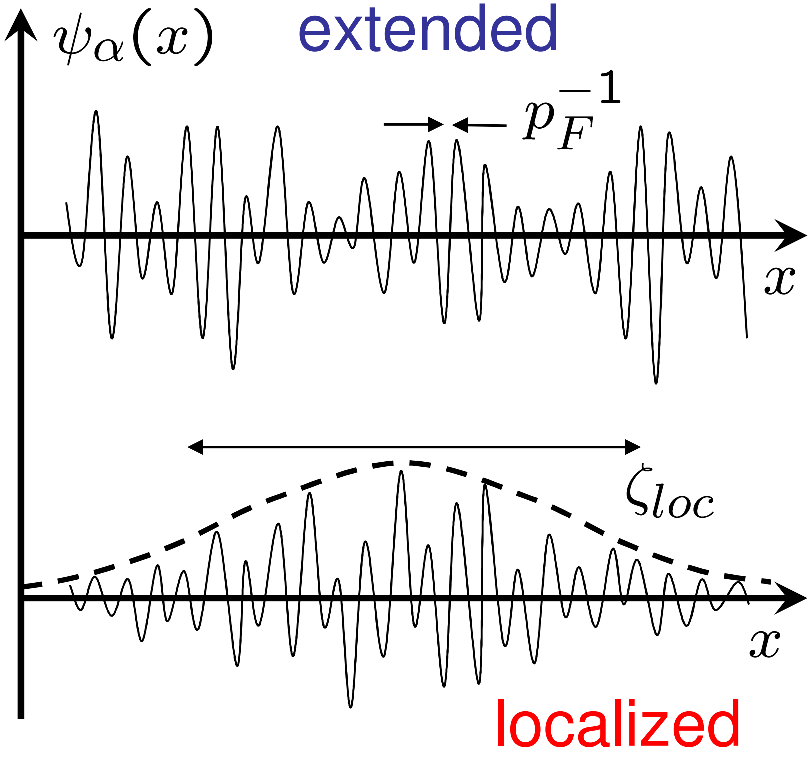
P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

1. Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha\psi_\alpha(\boldsymbol{r})$$



d=1; All states are *localized*

Exact solution for one channel:

M.E. Gertsenshtein, V.B. Vasil'ev, (1959)

"Conjecture" for one channel:

Sir N.F. Mott and W.D. Twose (1961)

Exact solution for $\sigma(\omega)$ for one channel:

V.L. Berezinskii, (1973)

Scaling argument for multi-channel :

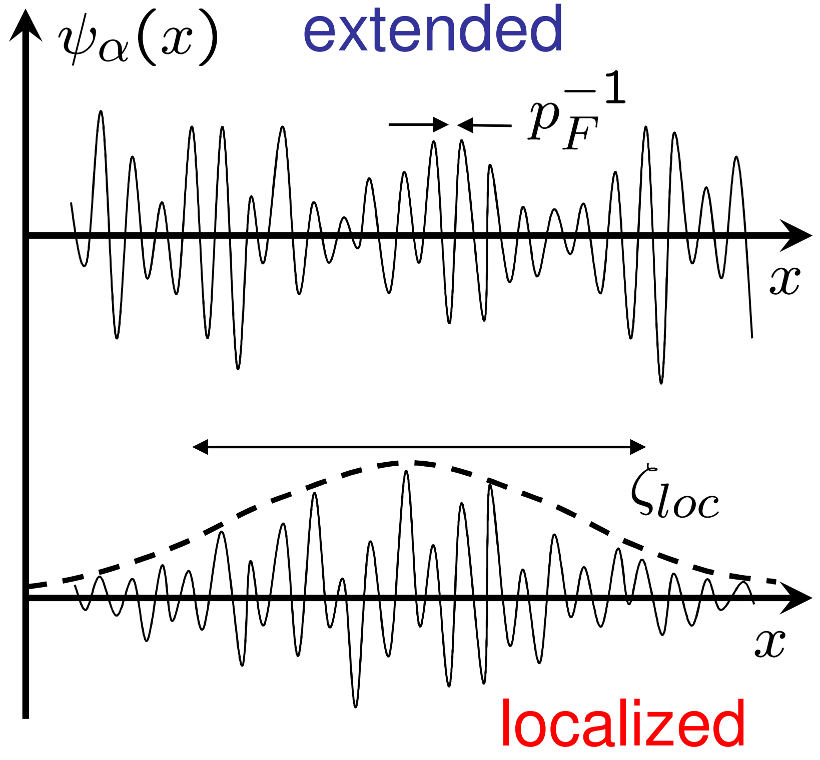
D.J. Thouless, (1977)

Exact solutions for multi-channel:

K.B.Efetov, A.I. Larkin (1983)
O.N. Dorokhov (1983)

1. Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



d=1; All states are *localized*

d=2; All states are *localized*
If no spin-orbit interaction

Thouless scaling + ansatz:

E. Abrahams, P. W. Anderson, D. C. Licciardello, and T.V. Ramakrishnan, (1979)

Instability of metal with respect to quantum (weak localization) corrections:
L.P. Gorkov, A.I.Larkin, D.E. Khmelnitskii, (1979)

First numerical evidence:
A Maccinnon, B. Kramer, (1981)

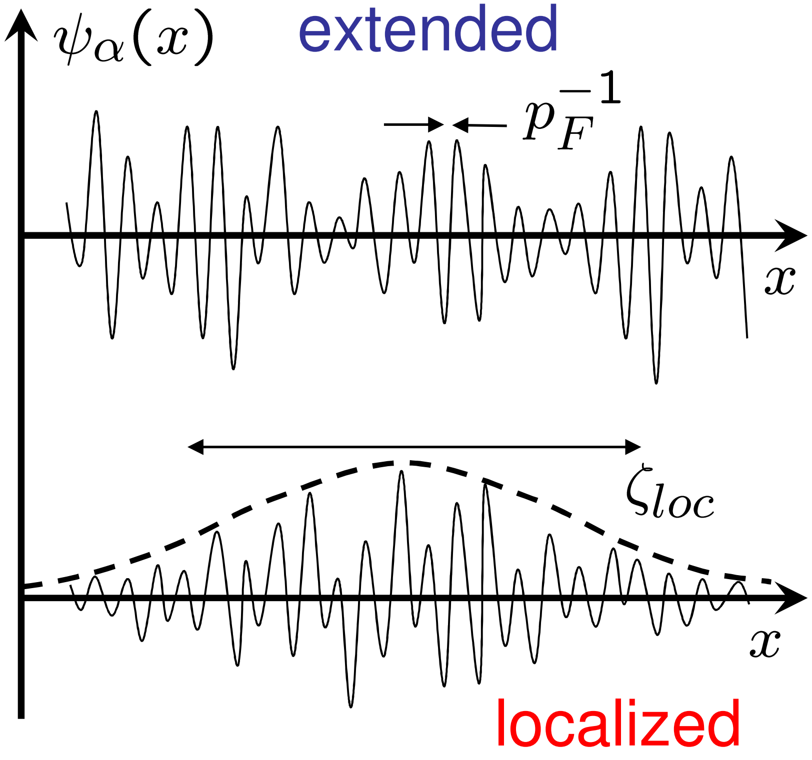
Instability of 2D metal with respect to quantum (weak localization) corrections:

L.P. Gorkov, A.I.Larkin, D.E. Khmelnitskii, (1979)

$$\sigma(\omega) = \sigma_D - \frac{e^2}{4\pi^2\hbar} \ln\left(\frac{1}{\omega\tau}\right)$$

1. Localization of single-electron wave-functions:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



d=1; All states are *localized*

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First numerical evidence:
A Maccinnon, B. Kramer, (1981)

“All states are ***localized*** “

means

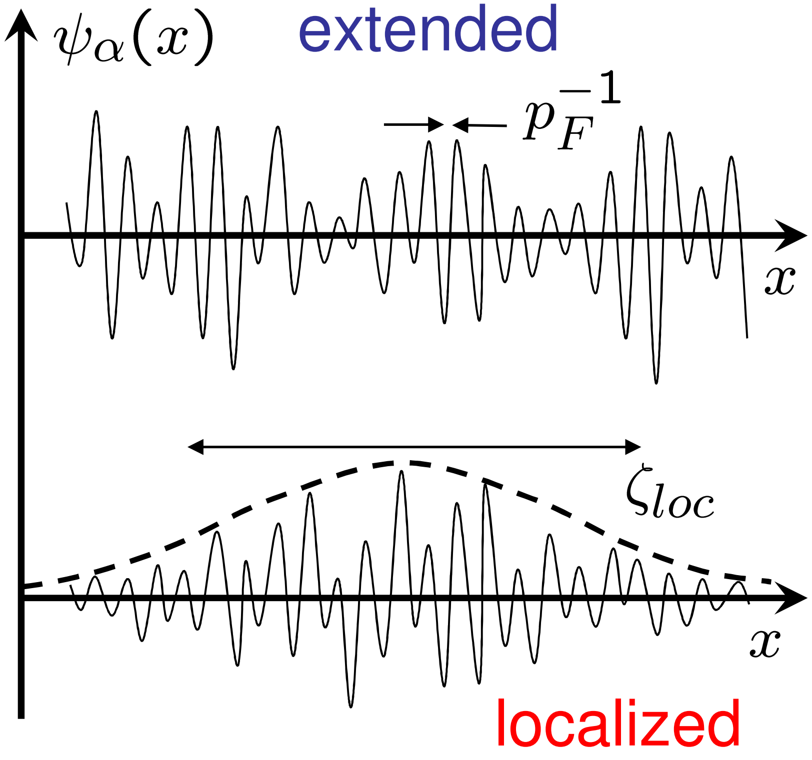
Probability to find an extended state:

$$\mathcal{P}_{ext} \propto \exp\left(-\#\frac{L}{\zeta_{loc}}\right)$$

System size

1. Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$

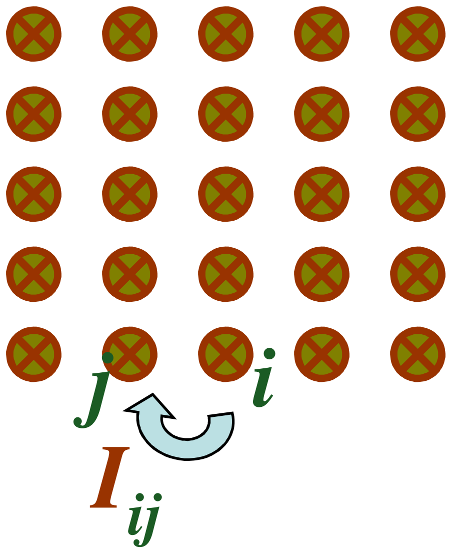


d=1; All states are ***localized***

d=2; All states are ***localized***

d>2; ***Anderson transition***

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* I_{ij}

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

$$-W < \varepsilon_i < W$$

uniformly distributed

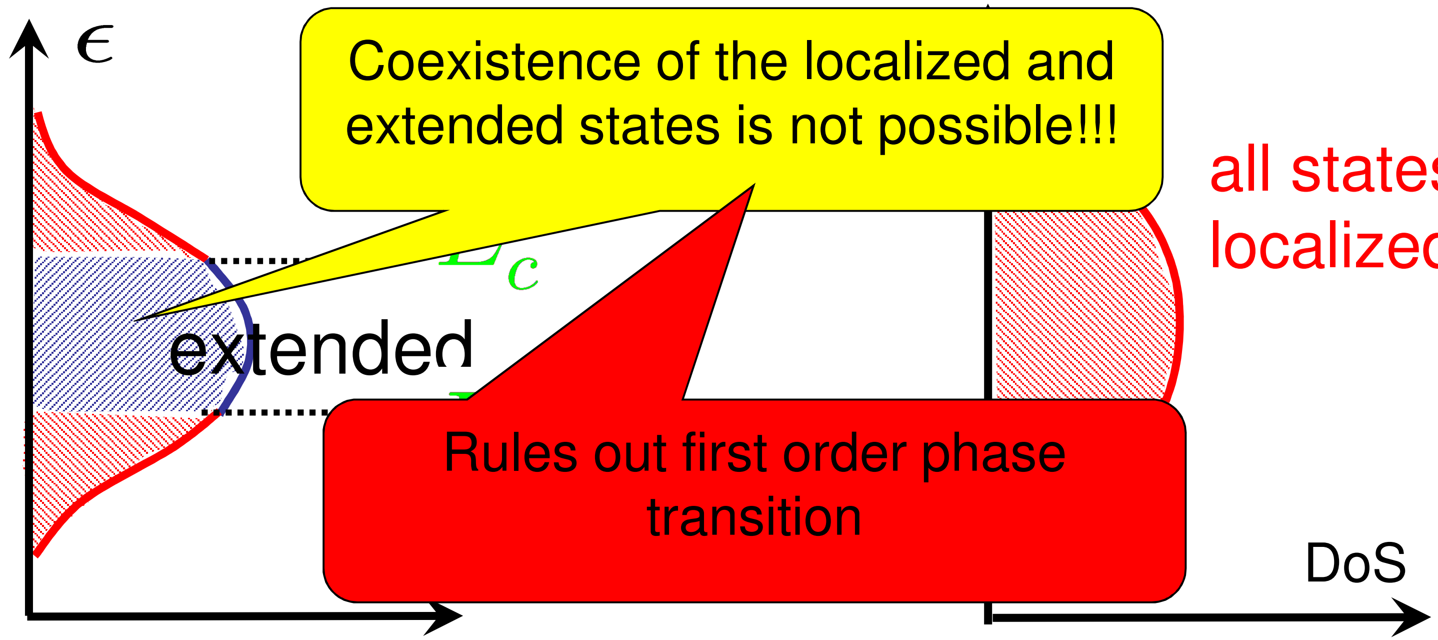
Critical hopping:

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)$$

$$d \gtrsim 3 \gg 1$$

Anderson Transition

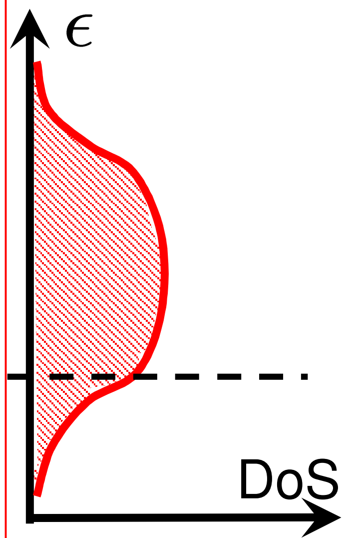
$I > I_c$ $\qquad$ $I < I_c$



E_c - mobility edges (one particle)

Temperature dependence of the conductivity (I)

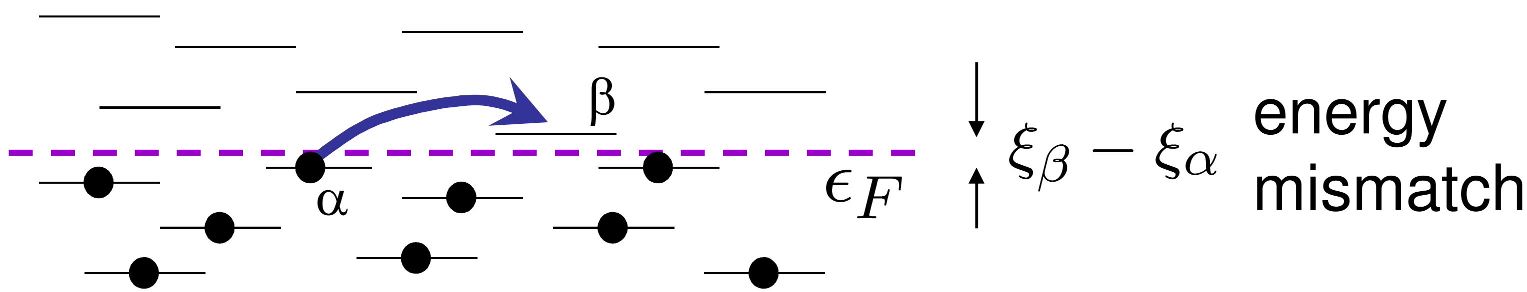
Assume that all the states are *localized*



F

$$\sigma(T) = 0$$

Inelastic processes) transitions between localized states

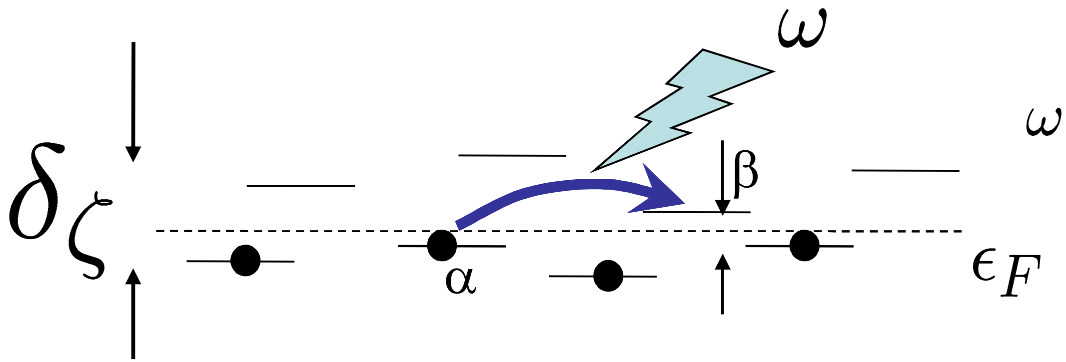


$\sigma(T) \propto \Gamma_\alpha$ (inelastic lifetime)$^{-1}$

$T = 0 \quad \Rightarrow \quad \sigma = 0$ (any mechanism)

$T > 0 \quad \Rightarrow \quad \sigma = ?$

Phonon-induced hopping



$$\omega = \xi_\beta - \xi_\alpha \ll \delta_\zeta$$

energy difference ca

Variable Range Hopping
Sir N.F. Mott (1968)

$$\sigma(T) \propto T^{\gamma} \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

Mechanism-dependent prefactor

Without omb gap
A.L.Efr

Optimized phase volume

Any bath with a continuous spectrum of delocalized excitations down to $\omega = 0$ will give the same exponential

Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

A#1: Sure

Easy steps: ***Person from the street (2005)***

1) Recall phonon-less AC conductivity:

Sir N.F. Mott (1970)

$$\sigma(\omega) \simeq \frac{e^2 \zeta_{loc}^{d-2}}{\hbar} \left(\frac{\hbar\omega}{\delta_\zeta}\right)^2 \ln^{d+1}\left|\frac{\delta_\zeta}{\hbar\omega}\right|$$

2) Calculate the Nyquist noise (fluctuation dissipation Theorem).

3) Use the electric noise instead of phonons.

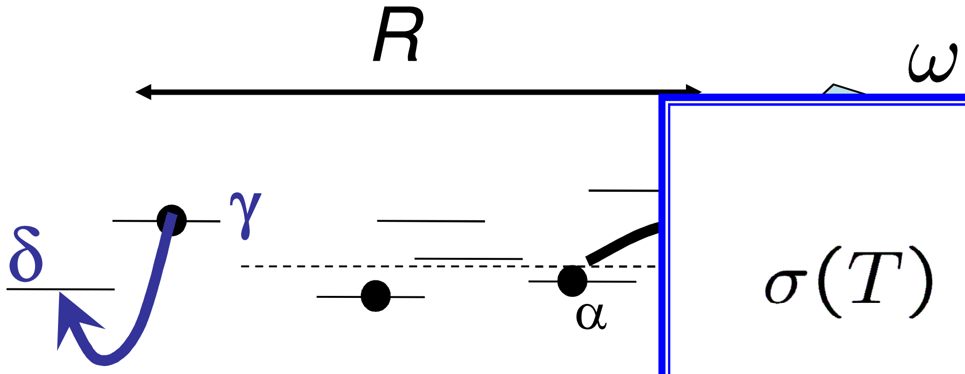
4) Do self-consistency (whatever it means).

Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

A#1: Sure ***[Person from the street (2005)]***

A#2: No way [L. Fleishman. P.W. Anderson (1980)]
(for Coulomb interaction in 3D – may be)

$R \to \infty$ ***Thus, the matrix element vanishes !!!***



$$\sigma(T) \propto 0 * \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

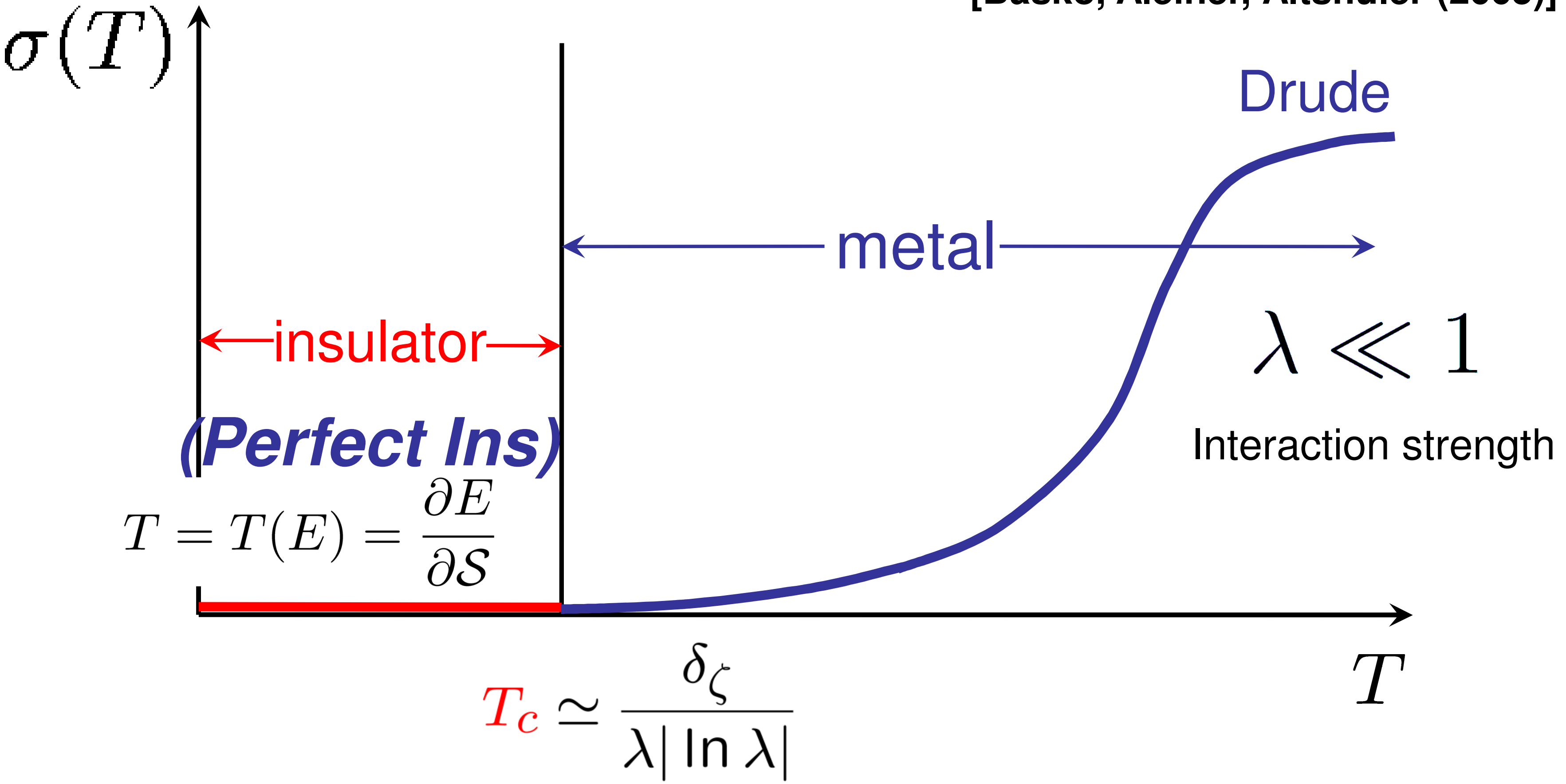
Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

A#1: Sure ***[Person from the street (2005)]***

A#2: No way [L. Fleishman. P.W. Anderson (1980)]

A#3: Finite T *Metal-Insulator Transition*

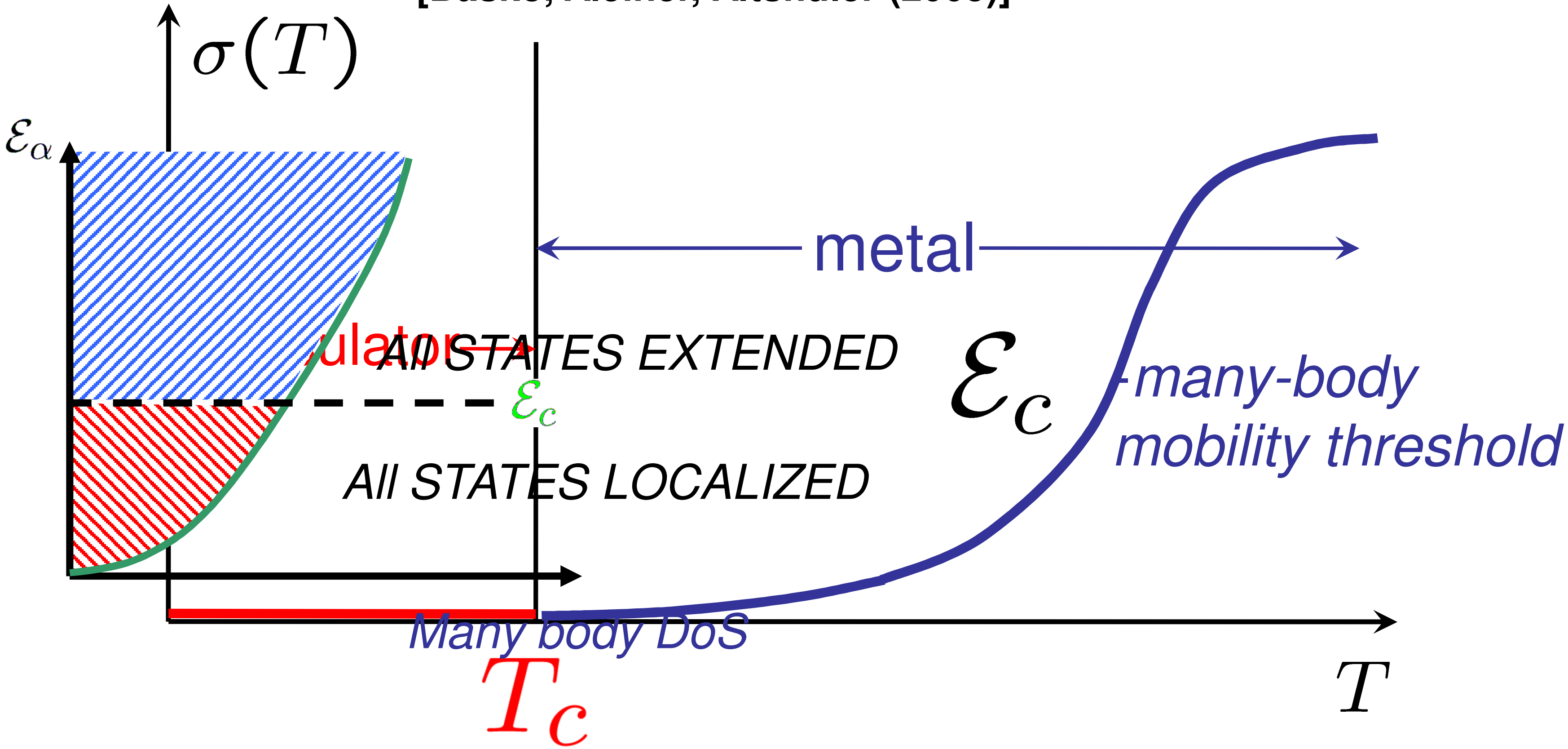
[Basko, Aleiner, Altshuler (2005)]



Many-body mobility threshold

$$\left[\hat{H}_1 + \hat{H}_{int}\right]\Psi_\alpha = \mathcal{E}_\alpha \Psi_\alpha$$

[Basko, Aleiner, Altshuler (2005)]



“All states are *localized* “

means

Probability to find an extended state:

$$\mathcal{P}_{ext} \propto \exp\left(-\#\frac{\mathcal{V}}{\mathcal{V}_{loc}(\mathcal{E})}\right)$$

$$\lim_{\mathcal{E}\to\mathcal{E}_c-0} \mathcal{V}_{loc}(\mathcal{E}) = \infty$$

System volume

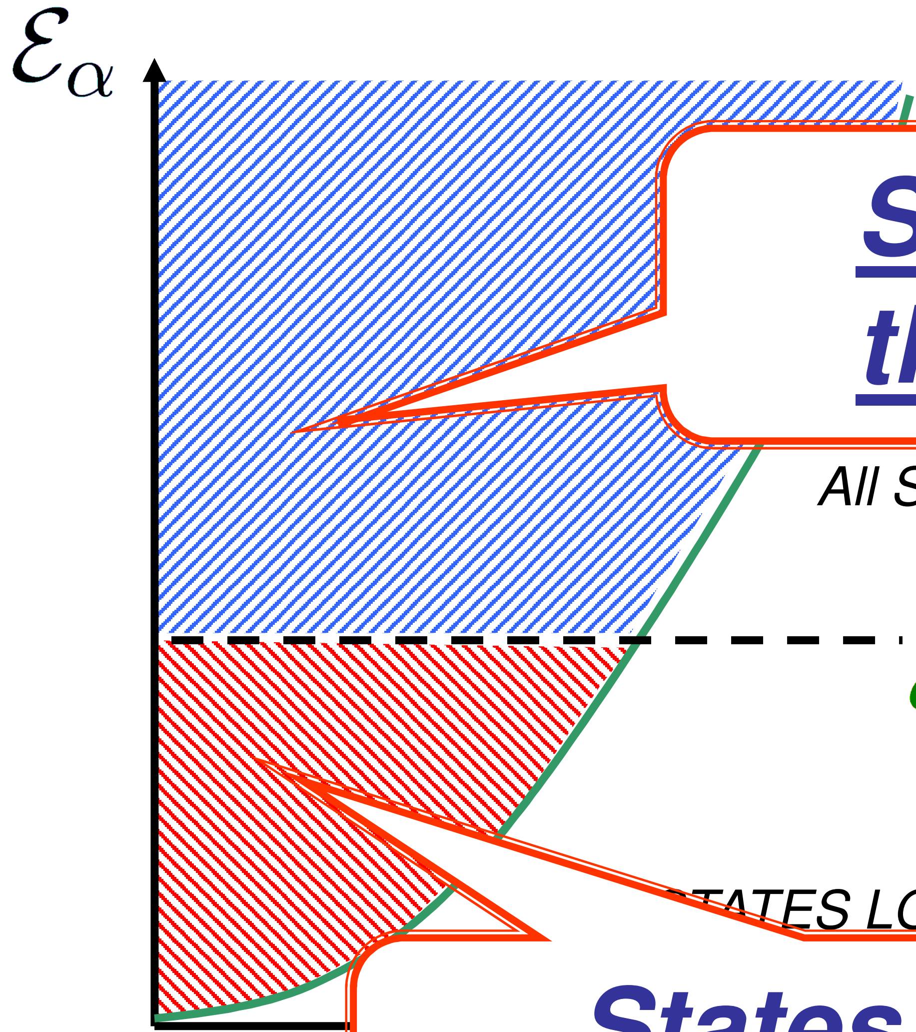
Localized one-body wave-function

Means, in particular:

$$\langle i\left|O(\boldsymbol{r}_1)\right|j\rangle\langle j\left|O(\boldsymbol{r}_2)\right|i\rangle \simeq \begin{cases} a\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{L(\omega)}\right), \quad \omega=\xi_i-\xi_j & \text{extended} \\ b\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\zeta_{loc}}\right), & \text{localized} \end{cases}$$

We define localized many-body wave-function as:

$$\langle \alpha\left|\hat{O}(\boldsymbol{r}_1)\right|\beta\rangle\langle \beta\left|\hat{O}(\boldsymbol{r}_2)\right|\alpha\rangle \simeq \begin{cases} \mathcal{A}\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{L(\omega)}\right), \quad \omega=\mathcal{E}_\alpha-\mathcal{E}_\beta & \text{extended} \\ \mathcal{B}\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\zeta_{loc}}\right), & \text{localized} \end{cases}$$



States always thermalized!!!

All STATES EXTENDED

$$\mathcal{E}_c = \int_0^{T_c} C_V(T)\,dT$$

TATES LOCALIZED

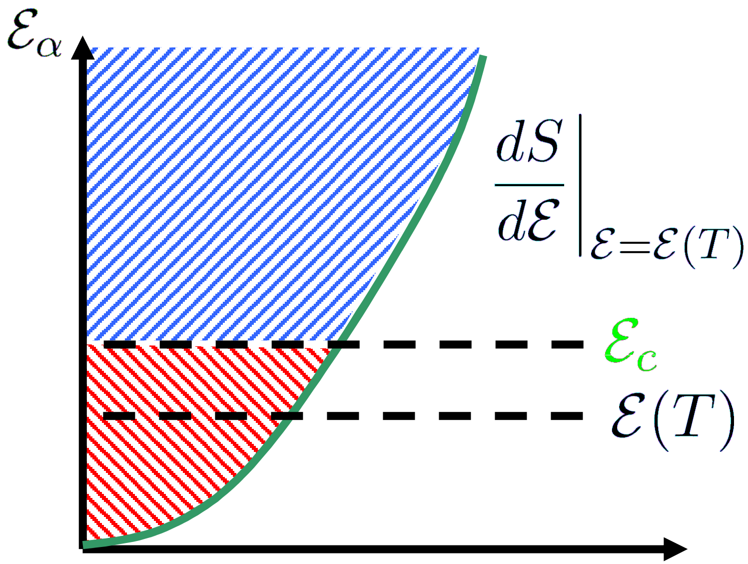
Entropy

States never thermalized!!!

Is it similar to Anderson transition?

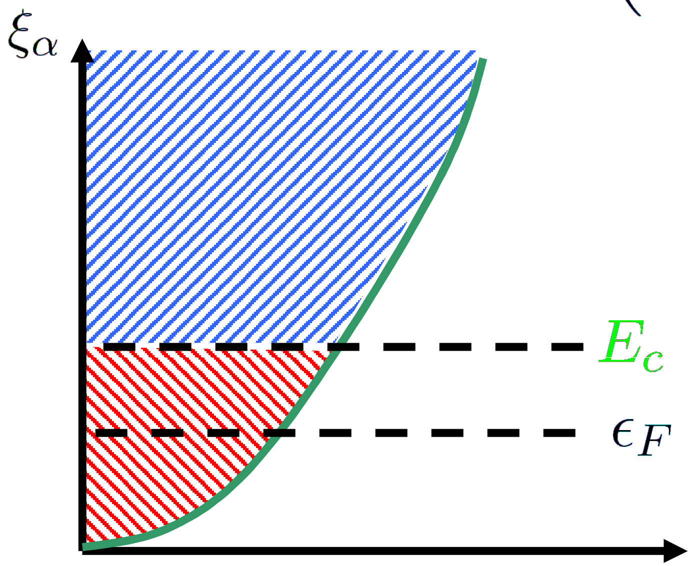
Why no activation?

$$\sigma(T) \propto \exp\left(-\frac{E_c - \epsilon_F}{T}\right)$$



$$\left.\frac{dS}{d\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}(T)} = \frac{1}{T}$$

Many body DoS



One-body DoS

$$\sigma(T) = \frac{\int_{\mathcal{E}_c}^{\infty} d\mathcal{E}\, e^{S(\mathcal{E})-\mathcal{E}/T}\sigma(\mathcal{E})}{\int_{0}^{\infty} d\mathcal{E}\, e^{S(\mathcal{E})-\mathcal{E}/T}} \simeq \exp\left[-\frac{1}{T}\int_{\mathcal{E}(T)}^{\mathcal{E}_c} \mathcal{E}\, d\mathcal{E}\, \frac{d^2 S}{d^2\mathcal{E}}\right] \xrightarrow[\mathcal{V}\to\infty]{} 0$$

$\propto \mathcal{V}$

$\sigma = 0$

Physics: Many-body excitations turn out to be localized in the Fock space

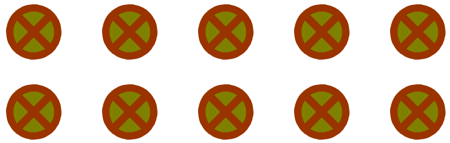
VOLUME 78, NUMBER 14 PHYSICAL REVIEW LETTERS 7 APRIL 1997

Quasiparticle Lifetime in a Finite System: A Nonperturbative Approach

Boris L. Altshuler,[1] Yuval Gefen,[2] Alex Kamenev,[2] and Leonid S. Levitov[3]
[1]*NEC Research Institute, 4 Independence Way, Princeton, New Jersey 08540*
[2]*Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot, 76100, Israel*
[3]*Massachusetts Institute of Technology, 12-112, Cambridge, Massachusetts 02139*
(Received 30 August 1996)

The problem of electron-electron lifetime in a quantum dot is studied beyond perturbation theory by mapping onto the problem of localization in the Fock space. Localized and delocalized regimes are identified, corresponding to quasiparticle spectral peaks of zero and finite width, respectively. In the localized regime, quasiparticle states are single-particle-like. In the delocalized regime, each eigenstate is a superposition of states with very different quasiparticle content. The transition energy is $\epsilon_c \simeq \Delta(g/\ln g)^{1/2}$, where Δ is mean level spacing, and g is the dimensionless conductance. Near ϵ_c there is a broad critical region not described by the golden rule. [S0031-9007(97)02895-0]

Anderson Model



• *Lattice - tight binding model*

In fact, i,j can be states in any space (not necessarily coordinate)

Critical hopping: I_{ij}

Interpretation:

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)$$

$$d \gtrsim 3 \gg 1$$

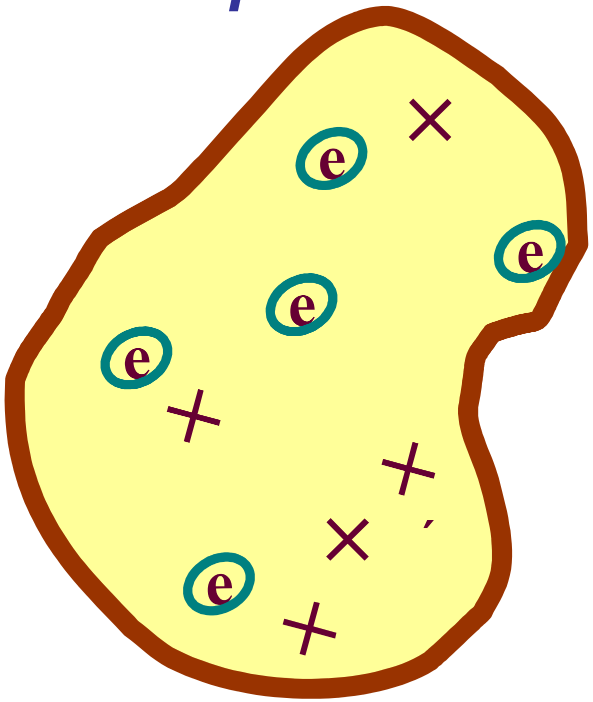
***W** – maximal energy mismatch;*

***2d** – number of coupled neighbors; (connectivity)*

At $I>I_c$ there will be always level mismatched from given by $|\varepsilon_i - \varepsilon_j| < I$

and the resonance transport will occur

Fock space localization in quantum dots (AGKL, 1997)



No spatial structure
(“0-dimensional”)

$$\hat{H} = \sum_{\alpha} \xi_\alpha \hat{c}^\dagger_\alpha \hat{c}_\alpha + \ldots + \lambda\delta_1 \sum_{\alpha\beta\gamma\delta} (\pm) \hat{c}^\dagger_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma \hat{c}_\delta$$

ξ_α -Random matrix theory

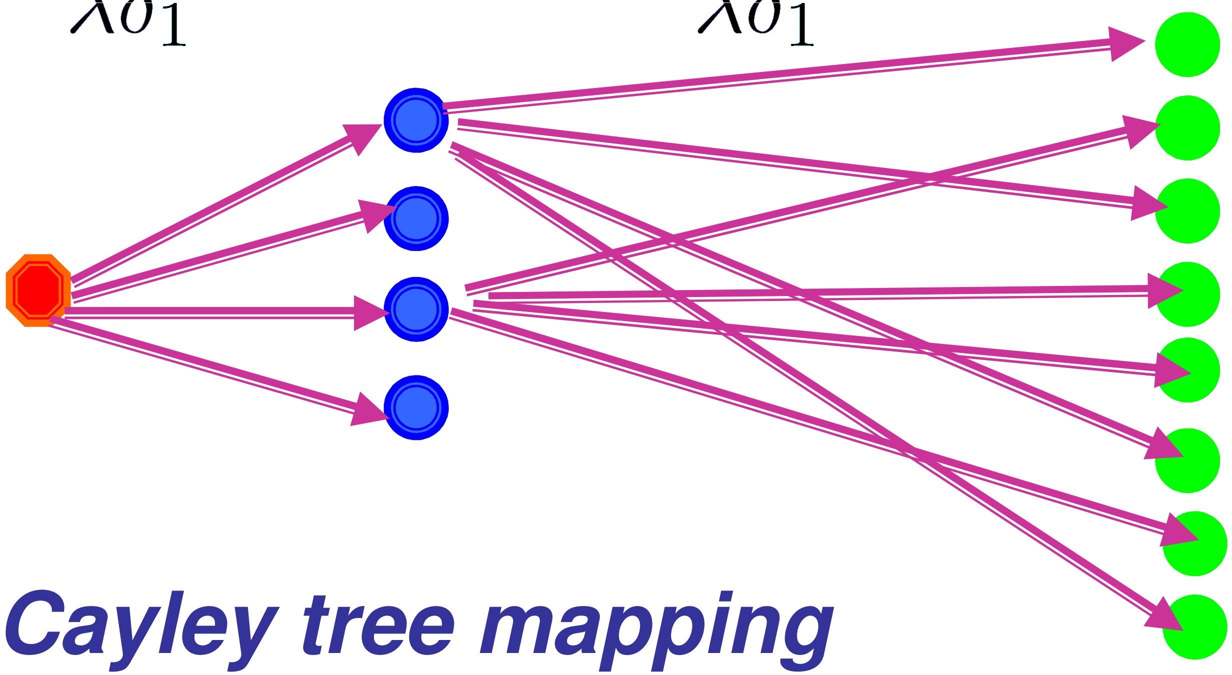
$\delta_1 = \langle \xi_{\alpha+1} - \xi_\alpha \rangle$ ***- one-particle level spacing;***

Fock space localization in quantum dots (AGKL, 1997)

$$\hat{H} = \sum_{\alpha} \xi_\alpha \hat{c}^\dagger_\alpha \hat{c}_\alpha + \ldots + \lambda\delta_1 \sum_{\alpha\beta\gamma\delta} (\pm) \hat{c}^\dagger_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma \hat{c}_\delta$$

***1*-particle excitation** | ***3*-particle excitation** | ***5*-particle excitation**

$$\xi_\alpha \xrightarrow[\lambda\delta_1]{} \xi_\gamma + \xi_\delta - \xi_\beta \xrightarrow[\lambda\delta_1]{} \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \ldots \quad \lambda\delta_1$$



Cayley tree mapping

Fock space localization in quantum dots (AGKL, 1997)

1-particle excitation	3-particle excitation	5-particle excitation

$$\xi_\alpha \xrightarrow[\lambda\delta_1]{} \xi_\gamma + \xi_\delta - \xi_\beta \xrightarrow[\lambda\delta_1]{} \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \ldots \quad \lambda\delta_1$$

$$(2d)\frac{I_c}{W} \simeq 1$$

$$\Downarrow$$

$$\left(\frac{T_c}{\delta_1}\right)^2 \lambda \simeq 1$$

$$I \rightarrow \lambda\delta_1$$

$$W \rightarrow \delta_1$$

$$2d \rightarrow \left(\frac{T}{\delta_1}\right)^2$$

δ_1 - ***one-particle level spacing;***

Metal-Insulator "Transition" in zero dimensions

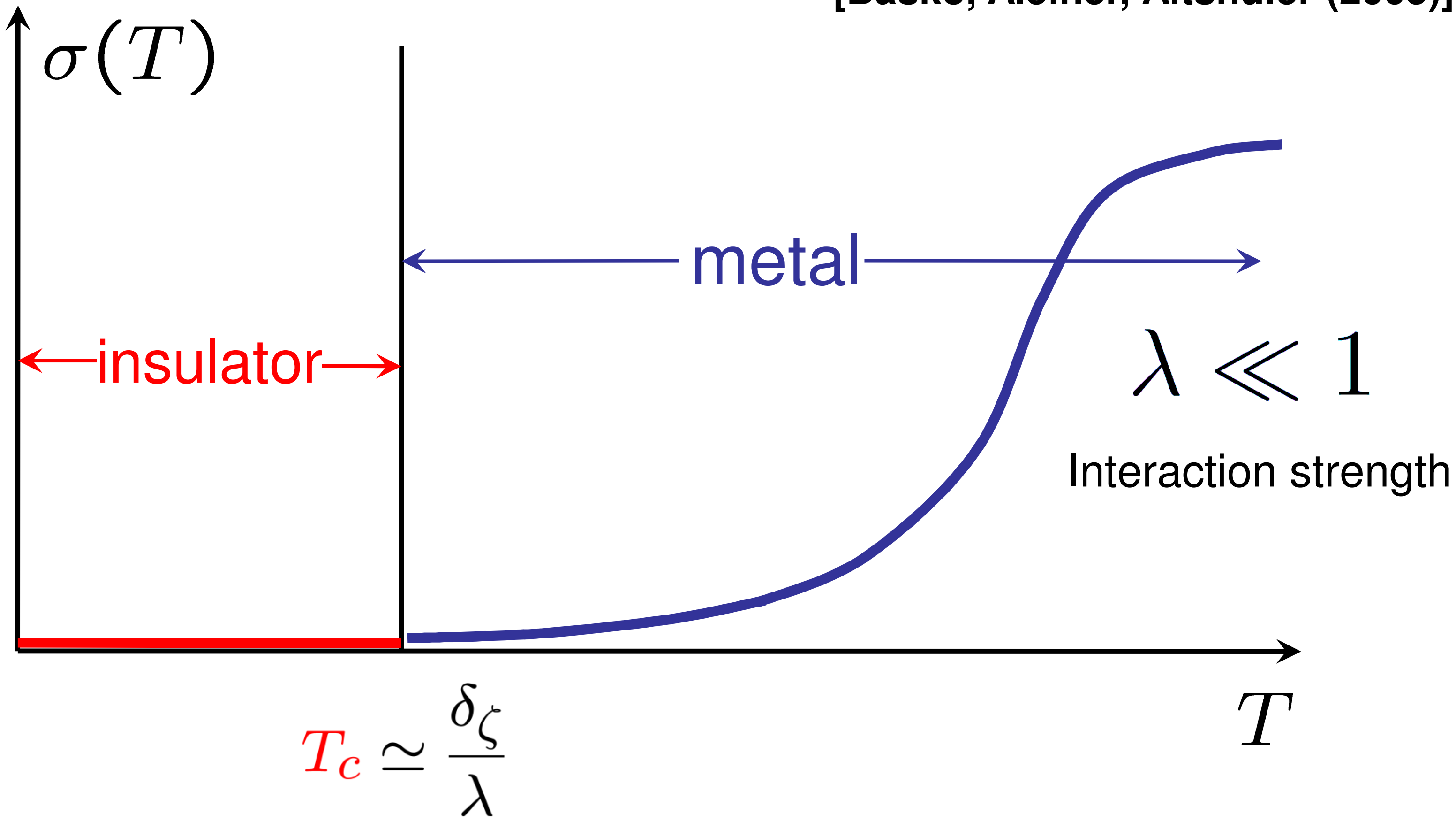
$$\left(\frac{T_c}{\delta_1}\right)^2 \simeq \frac{1}{\lambda}$$

[Altshuler, Gefen, Kamenev,Levitov (1997)]

In the paper: $\left(\frac{\epsilon_c}{\delta_1}\right)^2 \simeq \frac{1}{\lambda}\ln\frac{1}{\lambda}$

Vs. finite T *Metal-Insulator Transition in the bulk systems*

[Basko, Aleiner, Altshuler (2005)]



Metal-Insulator "Transition" in zero dimensions

[Altshuler, Gefen, Kamenev,Levitov (1997)]

$$\left(\frac{T_c}{\delta_1}\right)^2 \simeq \frac{1}{\lambda}$$

δ_1 - ***one-particle level spacing;***

Vs. finite T *Metal-Insulator Transition in the bulk systems*

[Basko, Aleiner, Altshuler (2005)]

$$T_c \simeq \frac{\delta_\zeta}{\lambda}$$

δ_ζ ***1-particle level spacing in localization volume;***

$$\delta_1 \longrightarrow \delta_\zeta$$

1) Localization in Fock space
= Localization in the coordinate space.

2) Interaction is local;

Metal-Insulator "Transition" in zero dimensions

[Altshuler, Gefen, Kamenev,Levitov (1997)]

$$\left(\frac{T_c}{\delta_1}\right)^2 \simeq \frac{1}{\lambda}$$

δ_1 - ***one-particle level spacing;***

Vs. finite T *Metal-Insulator Transition in the bulk systems*

[Basko, Aleiner, Altshuler (2005)]

$$T_c \simeq \frac{\delta_\zeta}{\lambda}$$

δ_ζ ***1-particle level spacing in localization volume;***

1,2) Locality:

$$\delta_1 \longrightarrow \delta_\zeta$$

3) Interaction matrix elements

$$\left(\frac{T}{\delta_\zeta}\right)^2 \longrightarrow \left(\frac{T}{\delta_\zeta}\right) \times \left(\frac{\omega}{\delta_\zeta}\right) \longrightarrow \left(\frac{T}{\delta_\zeta}\right) \times 1$$

Effective Hamiltonian for ***MIT***.

We would like to describe the low-temperature regime only.

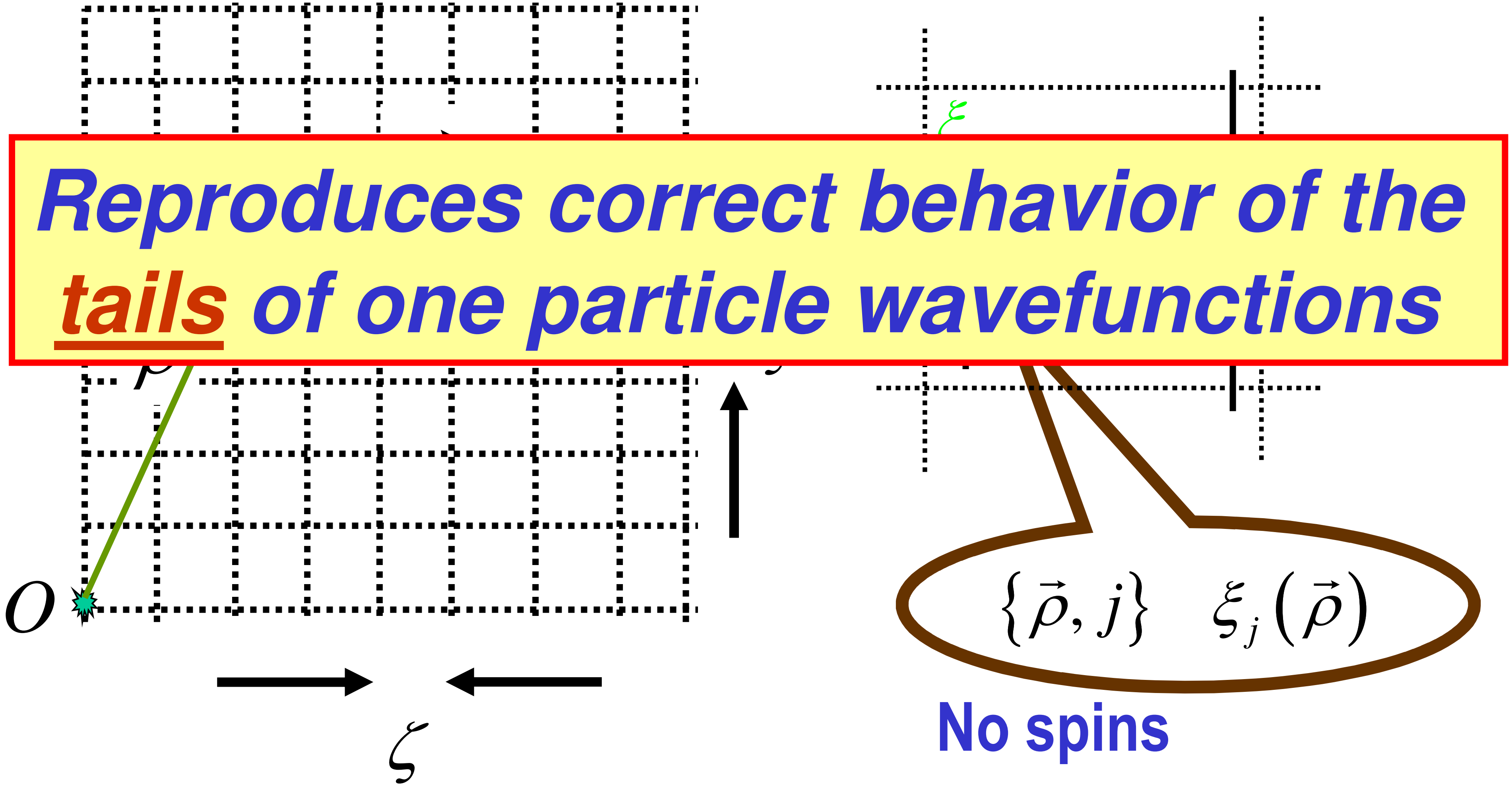
Spatial scales of interest >> ζ_{loc}

1-particle localization length

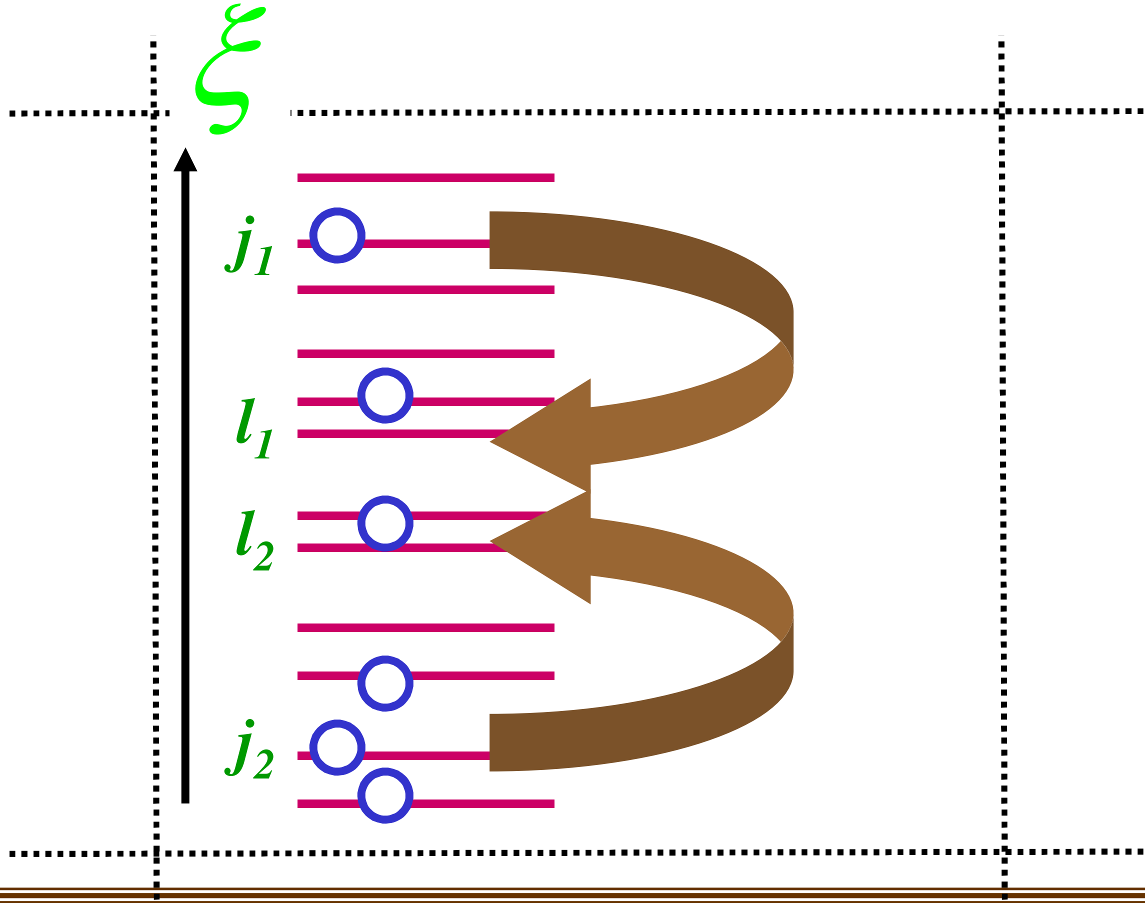
Otherwise, conventional perturbation theory for disordered metals works.

Altshuler, Aronov, Lee (1979); Finkelshtein (1983) – T-dependent SC potential
Altshuler, Aronov, Khmelnitskii (1982) – inelastic processes

Details: Seminar #1
December 26



$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$



$$\hat{V}_{int} = \frac{1}{2} \sum_{l_1 l_2 j_1 j_2; \boldsymbol{\rho}} V_{l_1 l_2}^{j_1 j_2}(\boldsymbol{\rho}) \hat{c}_{l_1}^{\dagger}(\boldsymbol{\rho}) \hat{c}_{l_2}^{\dagger}(\boldsymbol{\rho}) \hat{c}_{j_2}(\boldsymbol{\rho}) \hat{c}_{j_1}(\boldsymbol{\rho})$$

Interaction only within the same cell;

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

$$\hat{V}_{int} = \frac{1}{2} \sum_{l_1 l_2 j_1 j_2;\boldsymbol{\rho}} V_{l_1 l_2}^{j_1 j_2}(\boldsymbol{\rho}) \hat{c}_{l_1}^\dagger(\boldsymbol{\rho}) \hat{c}_{l_2}^\dagger(\boldsymbol{\rho}) \hat{c}_{j_2}(\boldsymbol{\rho}) \hat{c}_{j_1}(\boldsymbol{\rho})$$

Statistics of matrix elements?

Energy transfer $\omega \gg \delta_\zeta$

corresponds to the special scale $L_\omega = \sqrt{D/\omega} \ll \zeta$.

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \hat{c}_l^\dagger(\boldsymbol{\rho}) \left[\xi_l(\boldsymbol{\rho})\hat{c}_l(\boldsymbol{\rho}) + I\delta_\xi \sum_{\boldsymbol{a},m} \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

$$\hat{V}_{int} = \frac{1}{2} \sum_{l_1 l_2 j_1 j_2;\boldsymbol{\rho}} V_{l_1 l_2}^{j_1 j_2}(\boldsymbol{\rho}) \hat{c}_{l_1}^\dagger(\boldsymbol{\rho}) \hat{c}_{l_2}^\dagger(\boldsymbol{\rho}) \hat{c}_{j_2}(\boldsymbol{\rho}) \hat{c}_{j_1}(\boldsymbol{\rho})$$

$$V_{l_1 l_2}^{j_1 j_2} = \frac{\lambda\delta_\zeta \sigma_{l_1}^{j_1} \sigma_{l_2}^{j_2}}{2} \Upsilon\left(\frac{\xi_{j_1} - \xi_{l_1}}{\delta_\zeta}\right) \Upsilon\left(\frac{\xi_{j_2} - \xi_{l_2}}{\delta_\zeta}\right) - (l_1 \leftrightarrow l_2)$$

$$\Upsilon(x) = \theta\left(\frac{M}{2} - |x|\right); \quad 1 \ll M \lesssim \frac{1}{\sqrt{\lambda}}$$

Parameters: $\lambda, I, M^{-1} \ll 1$

σ_l^j **random signs**

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

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Parameters:

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σ_l^j **random signs**

Ensemble averaging over: $\xi_l(\rho); \sigma_i^j = \pm 1$

Level repulsion: Only within one cell.

Probability to find n levels in the energy interval of the width E:

$$P(n,E) = \frac{e^{-E/\delta_\zeta}}{n!} \left(\frac{E}{\delta_\zeta}\right)^n \exp\left[-F\left(\frac{n\delta_\zeta}{E}\right)\right]$$

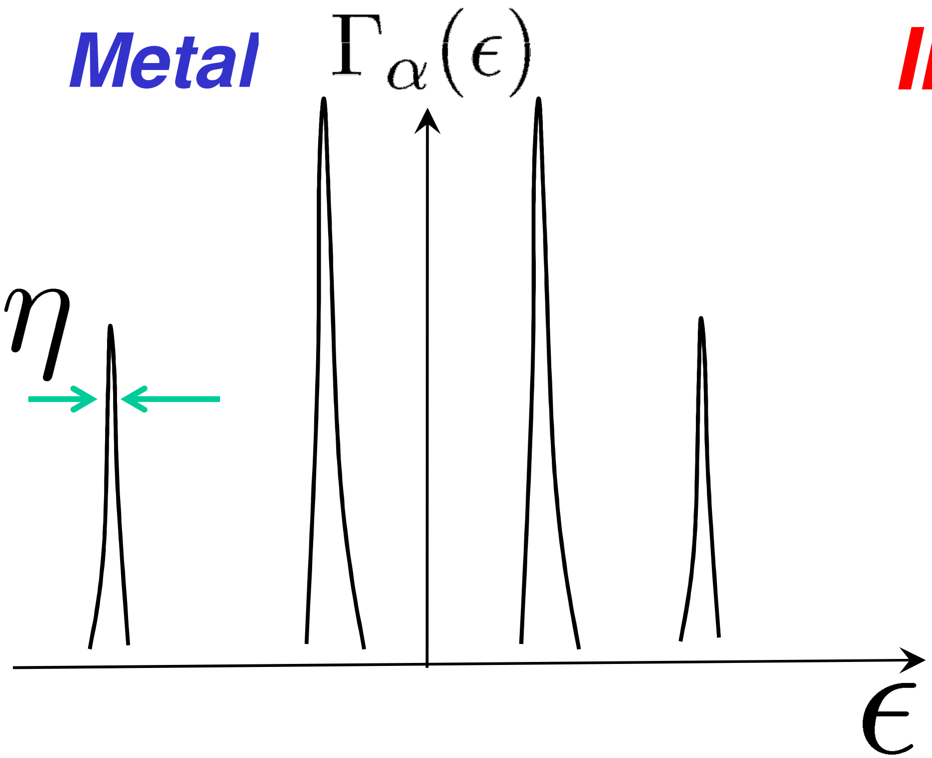
$$\lim_{x\to\infty} \frac{F(x)}{x} = \infty$$

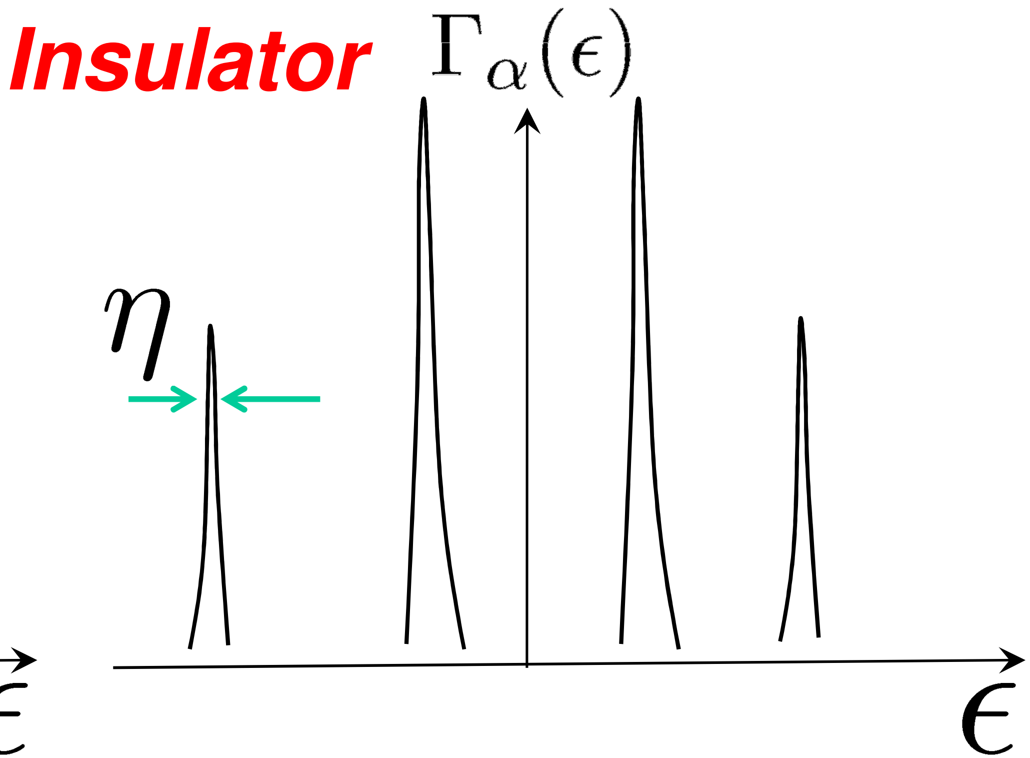
What to calculate?

Idea for one particle localization Anderson, (1958);
MIT for Cayley tree: Abou-Chakra, Anderson, Thouless (1973);
Critical behavior: Efetov (1987)

$$\Gamma_\alpha(\epsilon) = \mathrm{Im}\,\Sigma_\alpha^A(\epsilon)$$ – random quantity

No interaction: $\Gamma_\alpha(\epsilon) = \eta \to +0$





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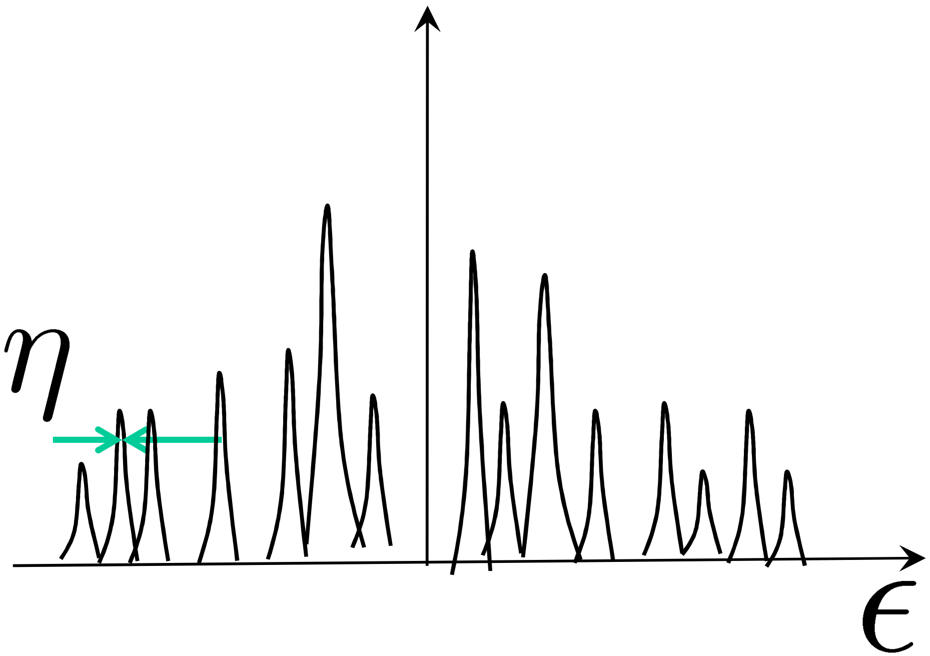
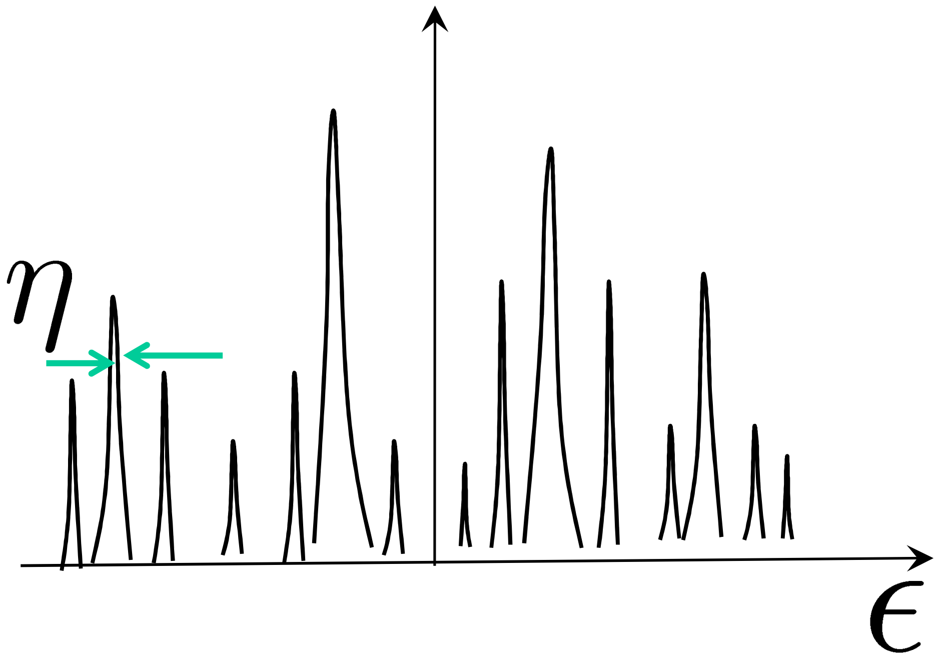
Metal $\Gamma_\alpha(\epsilon)$



Insulator $\Gamma_\alpha(\epsilon)$



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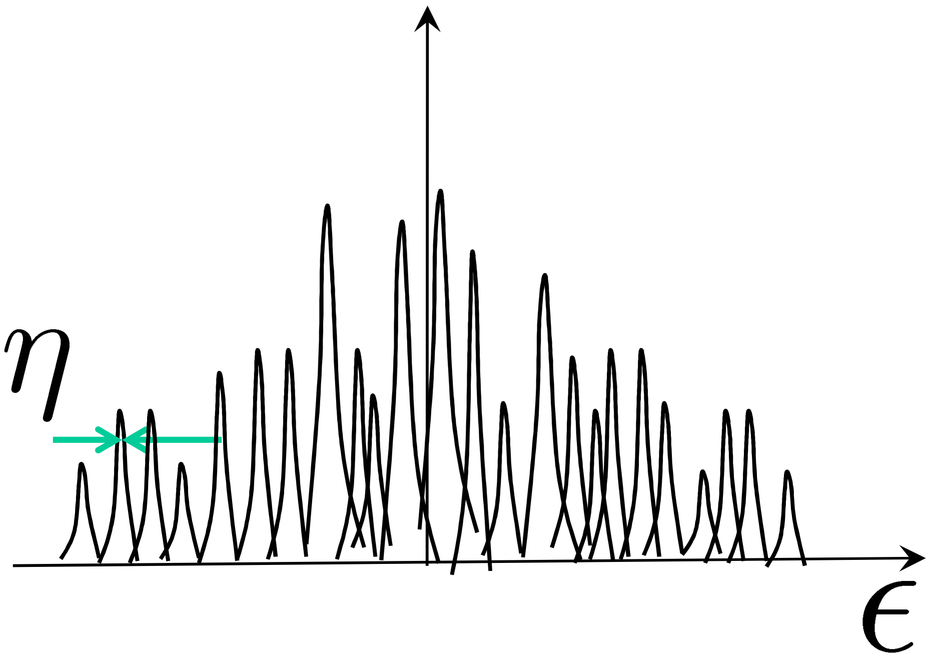

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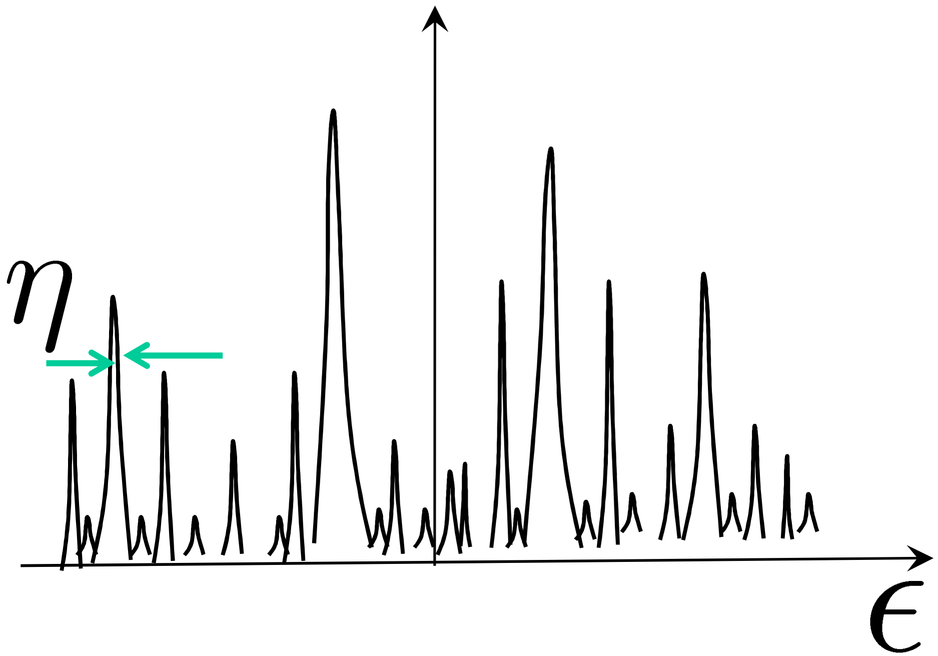

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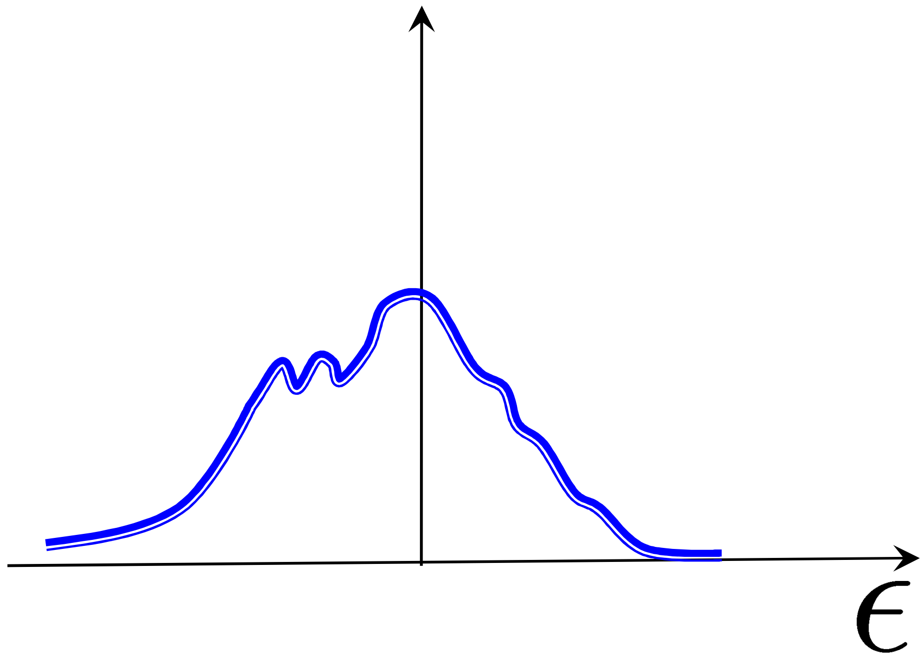
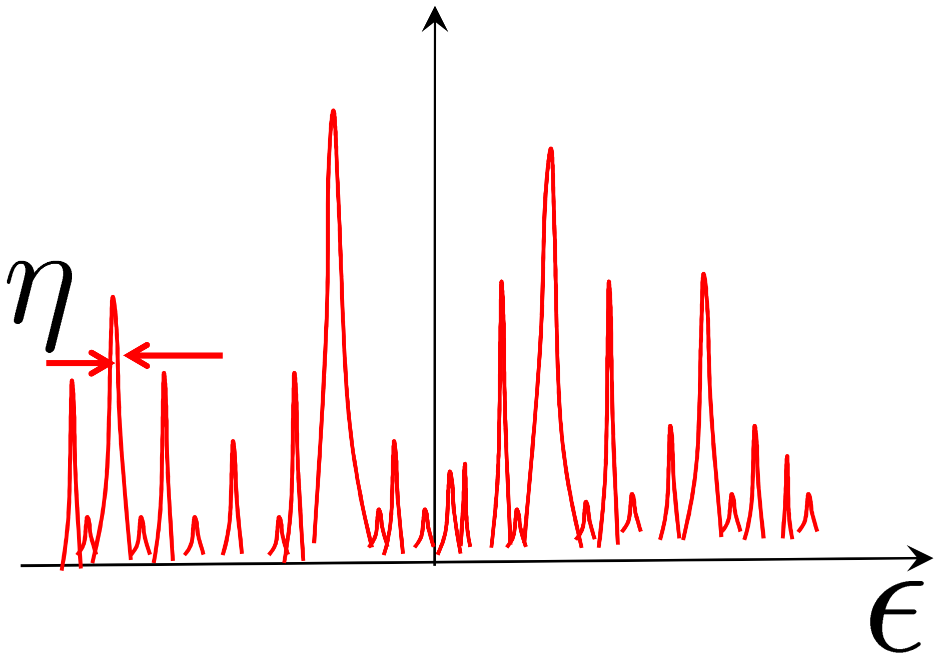


Insulator $\Gamma_\alpha(\epsilon)$



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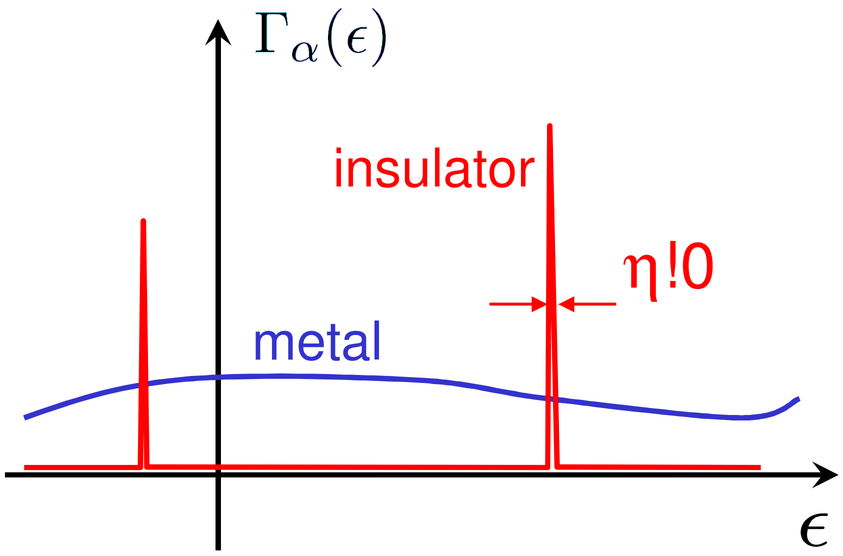
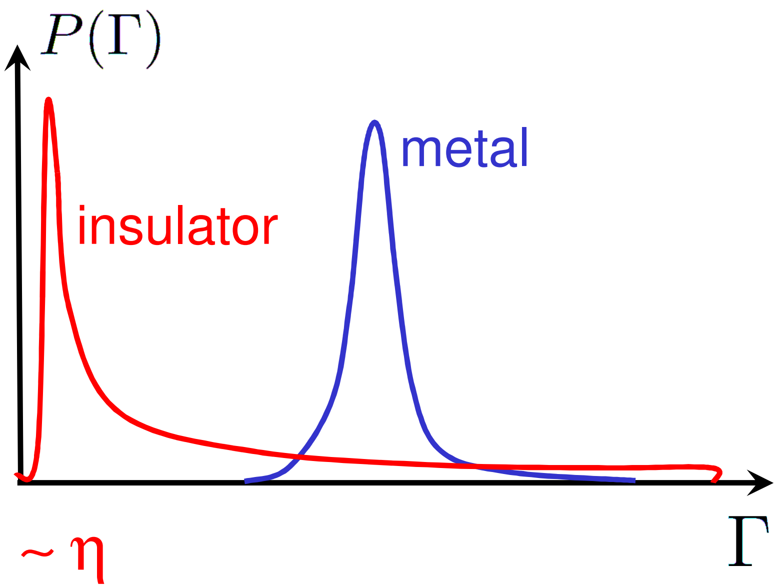
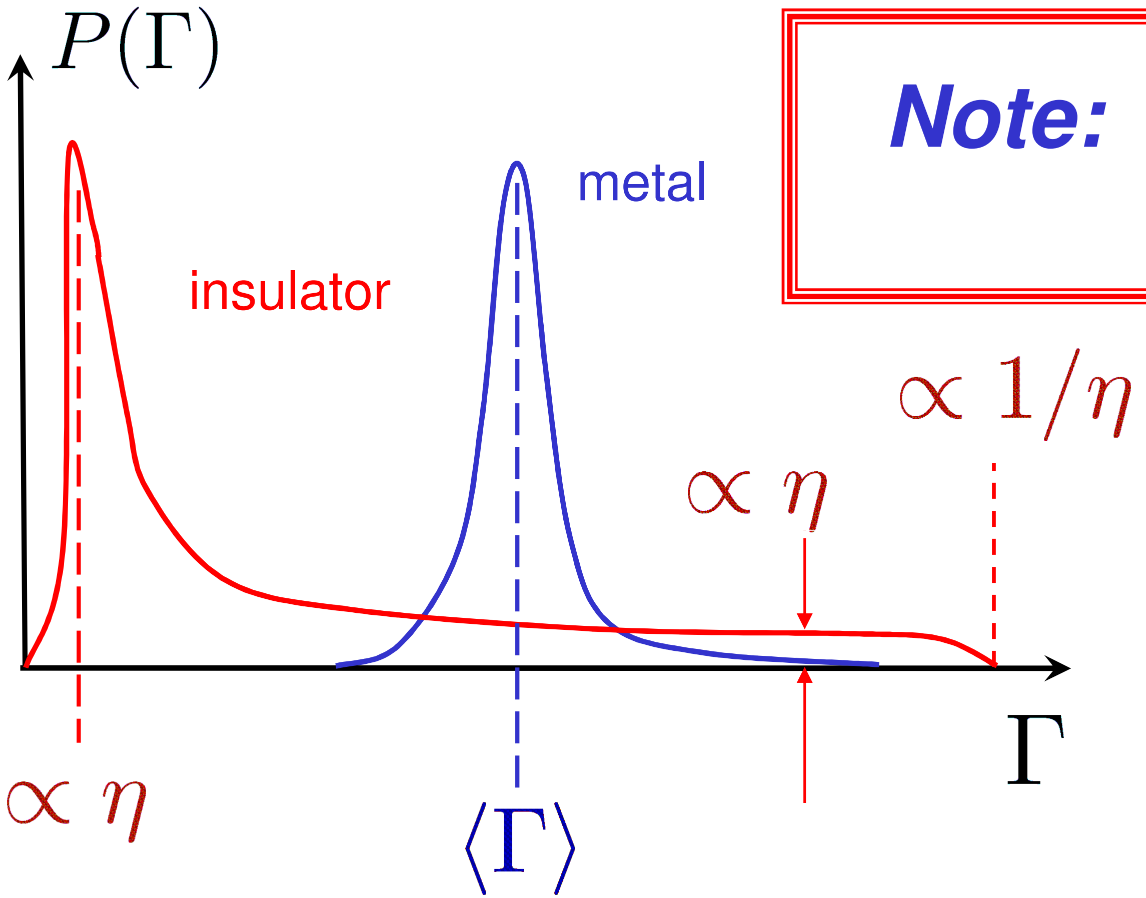


behavior for a given realization



probability distribution for a fixed energy

Probability Distribution



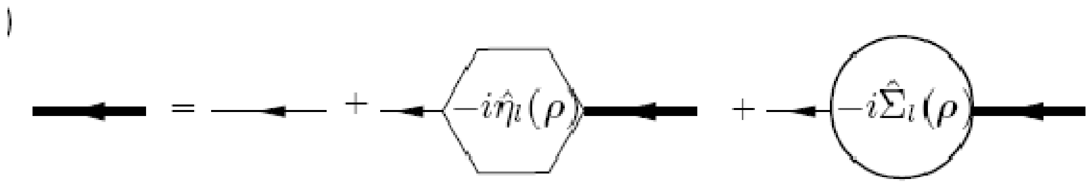
Note: $\langle\Gamma\rangle = \langle\Gamma\rangle$

Look for:

$$\lim_{\eta\to+0}\lim_{\mathcal{V}\to\infty} P(\Gamma>0) = \begin{cases} >0; & metal \\ 0; & insulator \end{cases}$$

How to calculate?

non-equilibrium (arbitrary occupations) → Keldysh

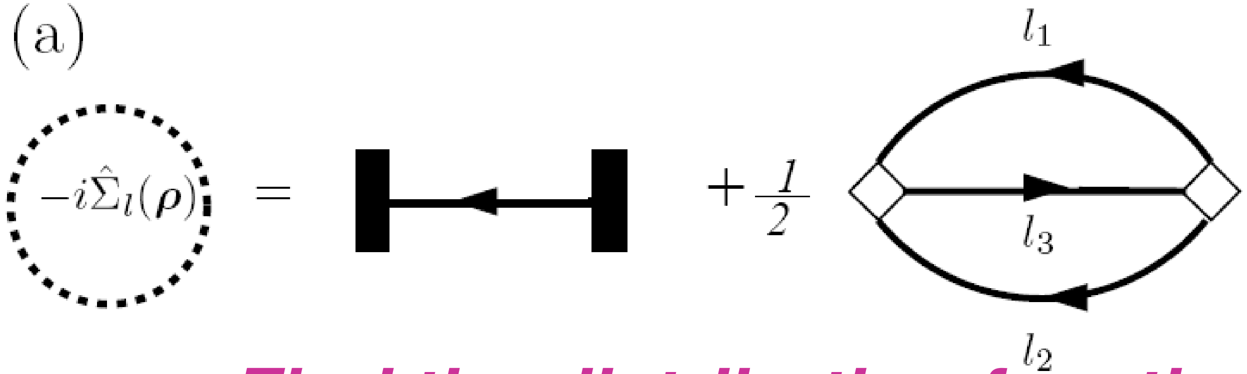


Parameters:

$$\lambda, I, M^{-1} \ll 1$$

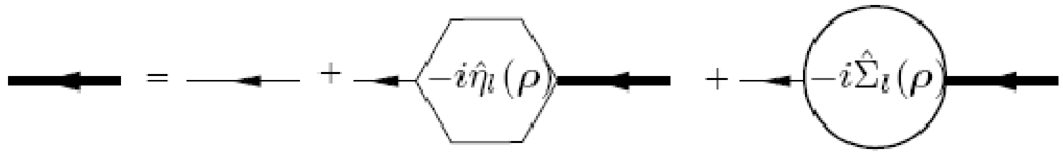
allow to select the most relevant series

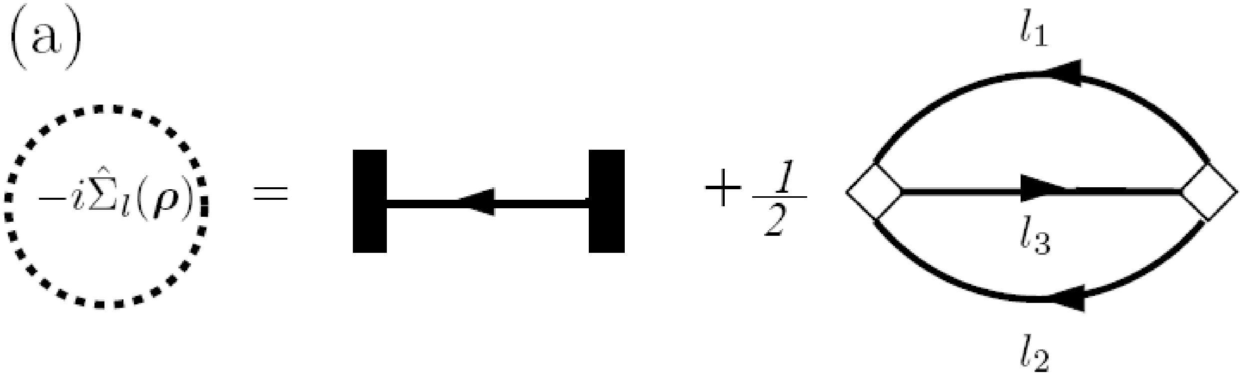
(a)



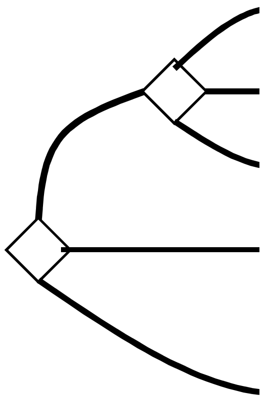
SCBA

Find the distribution function of each diagram





Iterations:



Cayley tree structure

Nonlinear integral equation with random coefficients

after standard simple tricks:

Decay due to tunneling

$$\Gamma_l(\epsilon) = \Gamma_l^{(el)}(\epsilon) + \Gamma_l^{(in)}(\epsilon) + \eta;$$

$$\Gamma_l^{(el)}(\epsilon, \boldsymbol{\rho}) = \pi I^2 \delta_\zeta^2 \sum_{l_1, \boldsymbol{a}} A_{l_1}(\epsilon, \boldsymbol{\rho} + \boldsymbol{a})$$

Decay due to e-h pair creation

$$\Gamma_l^{(in)}(\epsilon) = \pi \lambda^2 \delta_\zeta^2 \sum_{l_1, l_2, l_3} Y_{l_1, l_2}^{l_3, l} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 A_{l_1}(\epsilon_1) A_{l_2}(\epsilon_2) A_{l_3}(\epsilon_3) \delta(\epsilon - \epsilon_1 - \epsilon_2 + \epsilon_3) F^{\Rightarrow}_{l_1, l_2; l_3}(\epsilon_1, \epsilon_2; \epsilon_3);$$

$$A_l(\epsilon) = \frac{\pi^{-1}\Gamma_l(\epsilon)}{[\epsilon - \xi_l]^2 + [\Gamma_l(\epsilon)]^2}$$

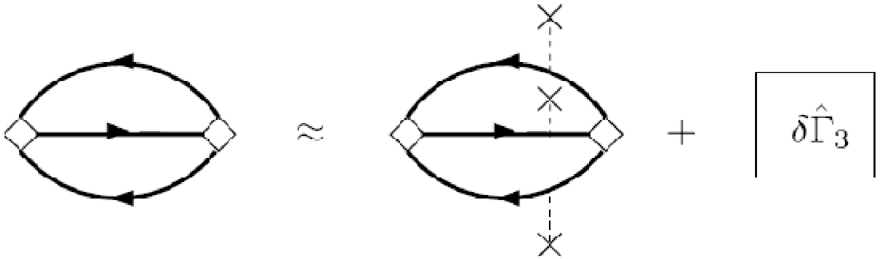
$$Y_{l_1, l_2}^{l_3, l} \equiv \frac{1}{2}\left[\Upsilon\left(\frac{\xi_{l_2} - \xi_l}{\delta_\zeta}\right)\Upsilon\left(\frac{\xi_{l_1} - \xi_{l_3}}{\delta_\zeta}\right) - \Upsilon\left(\frac{\xi_{l_1} - \xi_l}{\delta_\zeta}\right)\Upsilon\left(\frac{\xi_{l_2} - \xi_{l_3}}{\delta_\zeta}\right)\right]^2$$

$$F^{\Rightarrow}_{l_1, l_2; l_3}(\epsilon_1, \epsilon_2; \epsilon_3) = \frac{1}{4}\left\{1 + n_{l_1}(\epsilon_1) n_{l_2}(\epsilon_2) - n_{l_3}(\epsilon_3)\left[n_{l_1}(\epsilon_1) + n_{l_2}(\epsilon_2)\right]\right\};$$

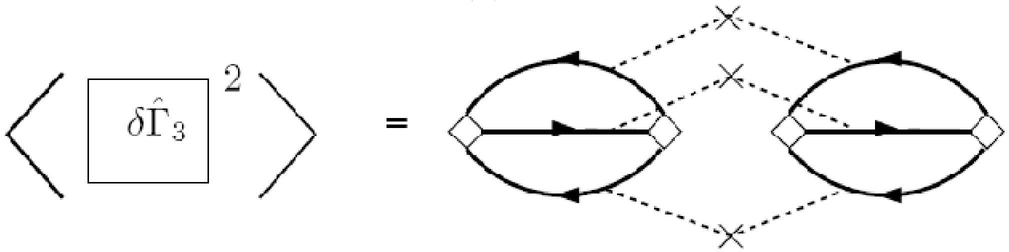
+ kinetic equation for occupation function $n_l(\epsilon)$

Stability of metallic phase

Assume $\Gamma_{in}(\epsilon)$ ***is Gaussian:***



a)



$$\left(\left\langle \Gamma^{(in)} \right\rangle = \pi \lambda^2 M T \right)^2$$

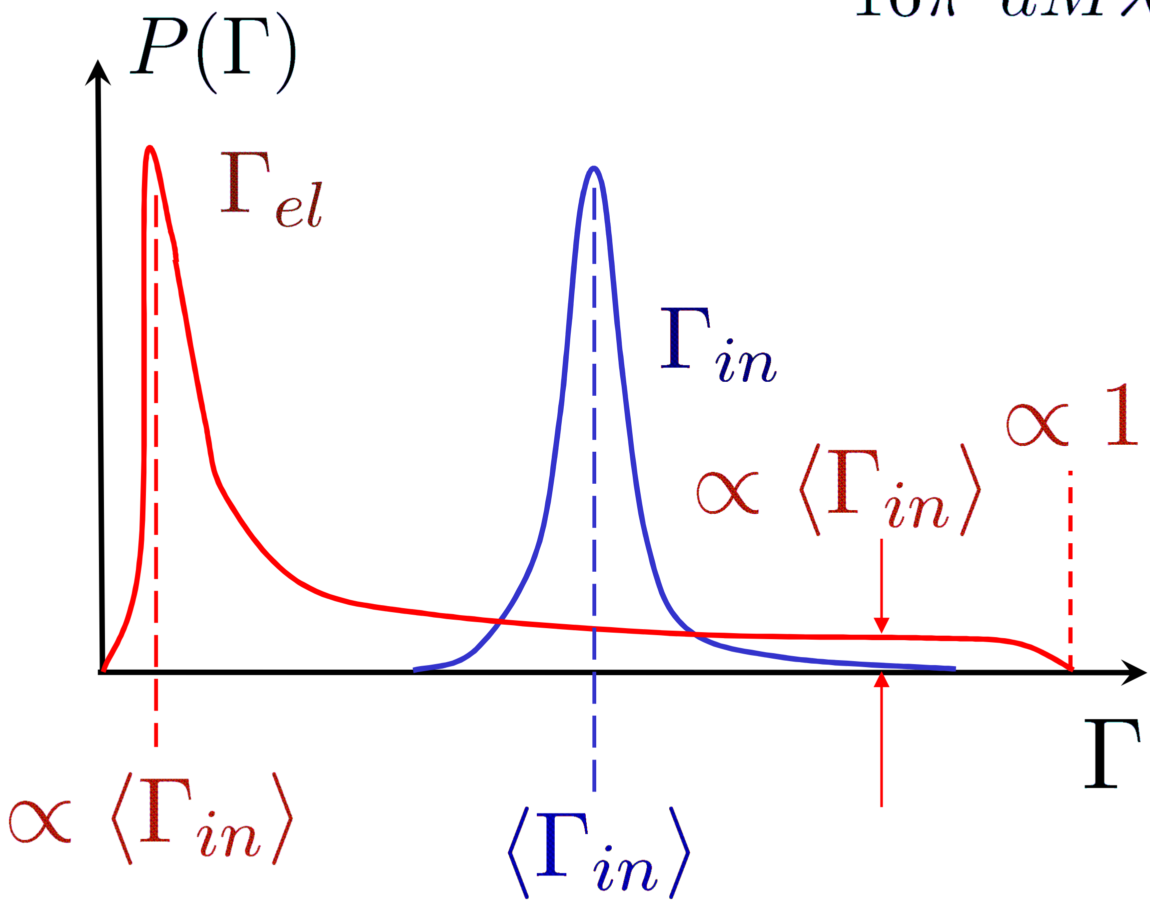
$$\Downarrow$$

$$\left\langle \left(\delta \Gamma^{(in)} \right)^2 \right\rangle = \frac{\pi \lambda^4 M \delta_\zeta^2 T}{36 \langle \Gamma^{(in)} \rangle}$$

$$T \gtrsim T_{in} \equiv \frac{\delta_\zeta}{6\pi\lambda M}$$

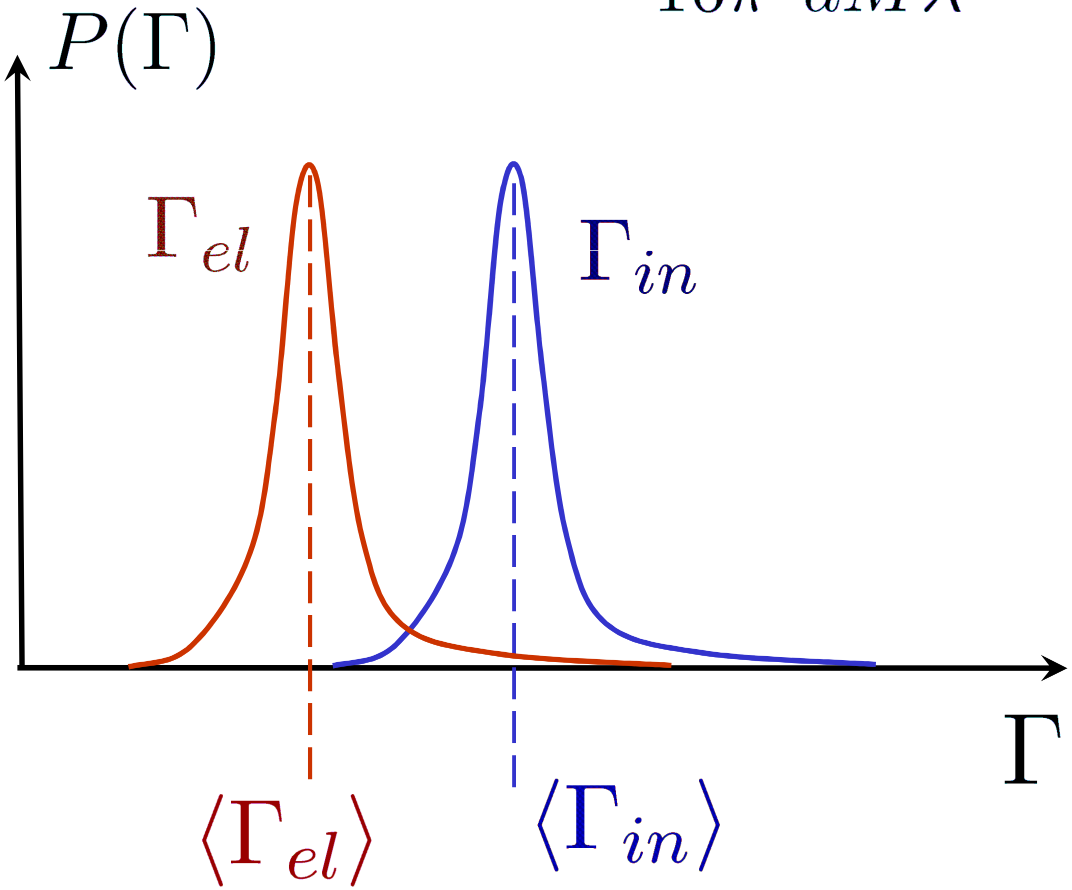
“Non-ergodic” metal [discussed first in AGKL,97]

$$T_{in} \lesssim T \lesssim T_{el} = \frac{\delta_\zeta}{16\pi^2 d M \lambda^2} \simeq \frac{T_{in}}{\lambda}$$



Drude metal

$$T \gtrsim T_{el} = \frac{\delta_\zeta}{16\pi^2 d M \lambda^2} \simeq \frac{T_{in}}{\lambda}$$

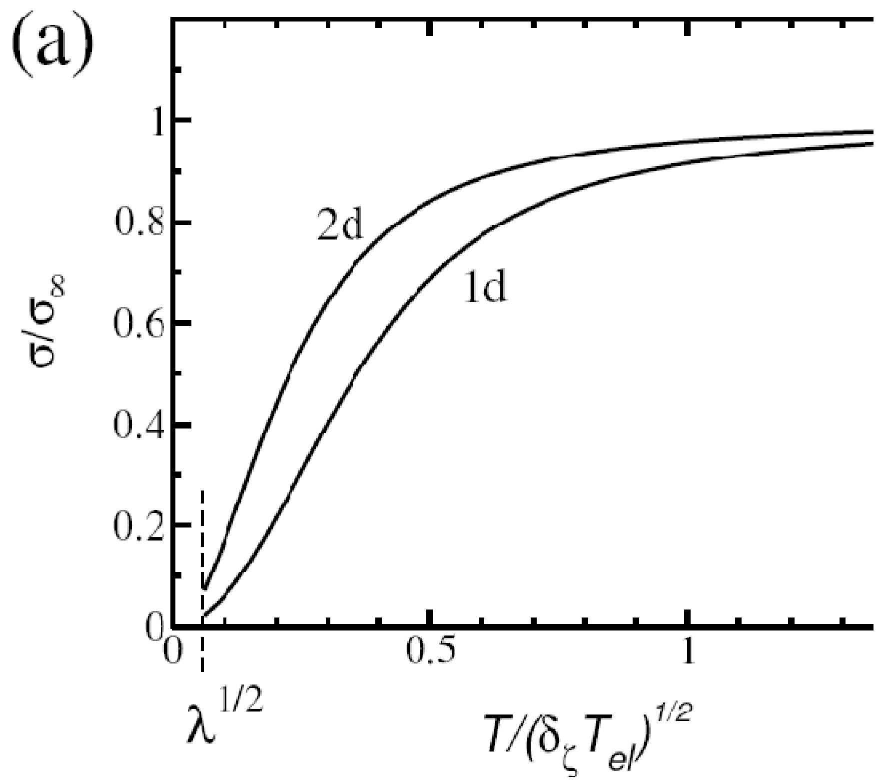


Kinetic Coefficients in Metallic Phase

$$\sigma_\infty \equiv \frac{2\pi e^2 I^2 \zeta_{loc}^{2-d}}{\hbar}$$

$$\sigma(T \gg \sqrt{\delta_\zeta T_{el}}) \approx \sigma_\infty \left(1 - \frac{2}{3}\,\frac{\delta_\zeta T_{el}}{T^2}\right)$$

$$\sigma(T \ll \sqrt{\delta_\zeta T_{el}}) = \sigma_\infty \frac{\pi}{4}\left(\frac{T^2}{\delta_\zeta T_{el}}\right)$$

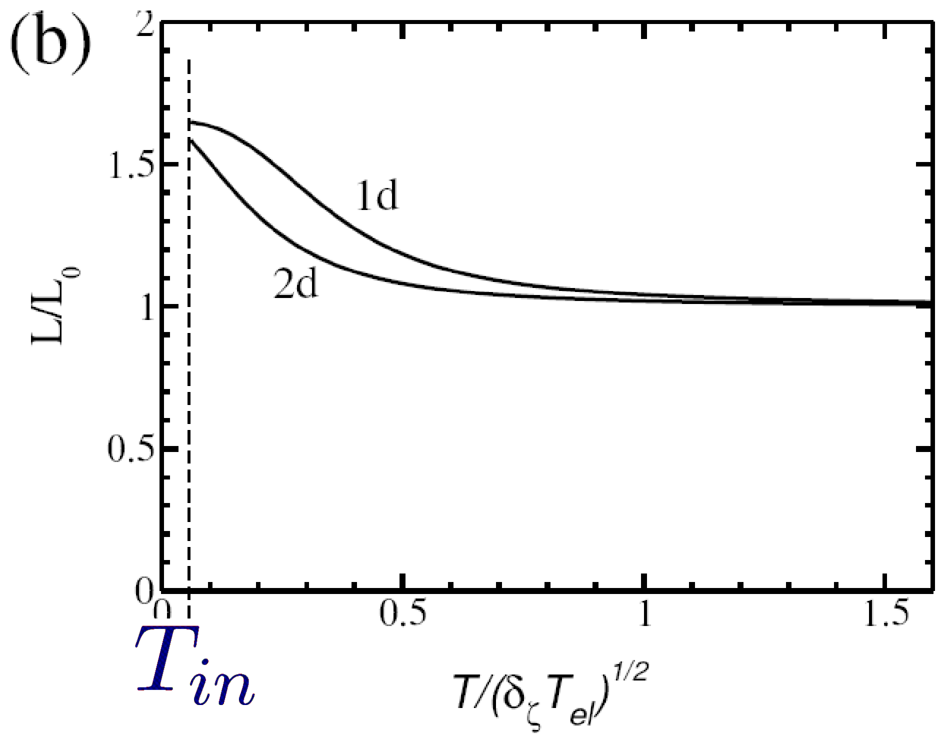


T_{in}

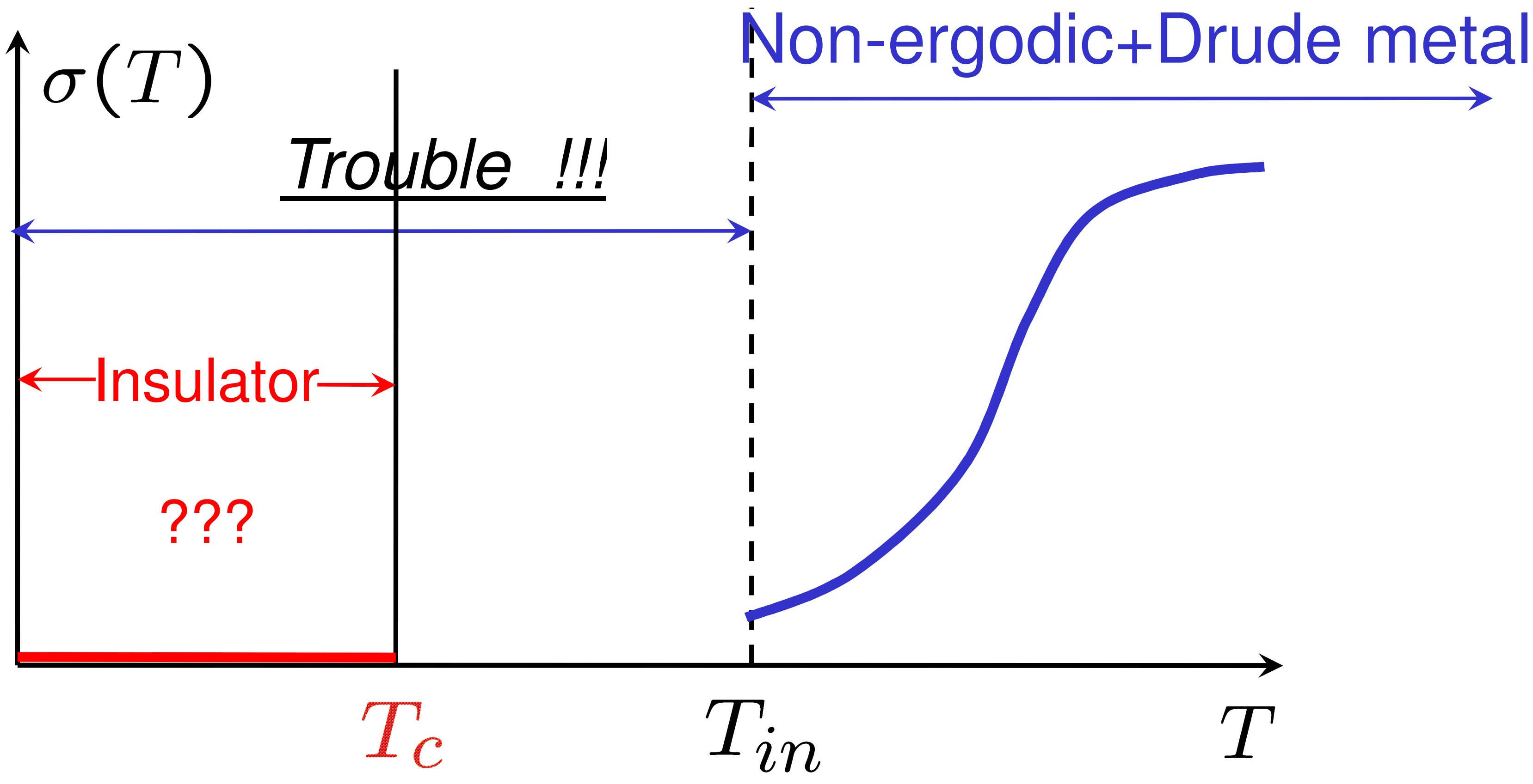
Kinetic Coefficients in Metallic Phase

Wiedemann-Frantz law ?

$$\frac{\mathrm{L}(T)}{\mathrm{L}_0} \equiv \frac{3e^2\kappa(T)}{\pi^2\sigma(T)T} = \begin{cases} 1 + 0.3\left(\frac{\delta_\zeta T_{el}}{T^2}\right), & T \gg \sqrt{\delta_\zeta T_{el}}, \\ \frac{192\mathrm{G}^2}{\pi^4} \approx 1.65\ldots, & T \ll \sqrt{\delta_\zeta T_{el}}. \end{cases}$$



So far, we have learned:



Stability of the insulator

Nonlinear integral equation with *random* coefficients

$$\Gamma_l(\epsilon) = \Gamma_l^{(el)}(\epsilon) + \Gamma_l^{(in)}(\epsilon) + \eta;$$

$$\Gamma_l^{(el)}(\epsilon, \boldsymbol{\rho}) = \pi I^2 \delta_\zeta^2 \sum_{l_1, \boldsymbol{a}} A_{l_1}(\epsilon, \boldsymbol{\rho} + \boldsymbol{a});$$

$$\Gamma_l^{(in)}(\epsilon) = \pi \lambda^2 \delta_\zeta^2 \sum_{l_1, l_2, l_3} Y_{l_1, l_2}^{l_3, l} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 A_{l_1}(\epsilon_1) A_{l_2}(\epsilon_2) A_{l_3}(\epsilon_3) \delta(\epsilon - \epsilon_1 - \epsilon_2 + \epsilon_3) F^{\Rightarrow}_{l_1, l_2; l_3}(\epsilon_1, \epsilon_2; \epsilon_3);$$

$$A_l(\epsilon) = \frac{\pi^{-1}\Gamma_l(\epsilon)}{[\epsilon + \xi_l]^2 + [\Gamma_l(\epsilon)]^2}$$

Notice: $\Gamma(\epsilon) = 0;$ **for** $\eta = 0$ **is a solution**

Linearization:

$$A_l(\epsilon) \approx \delta(\epsilon - \xi_l) + \frac{\Gamma_l(\epsilon)}{\pi(\epsilon - \xi_l)^2}$$

# of interactions

# of hops in space

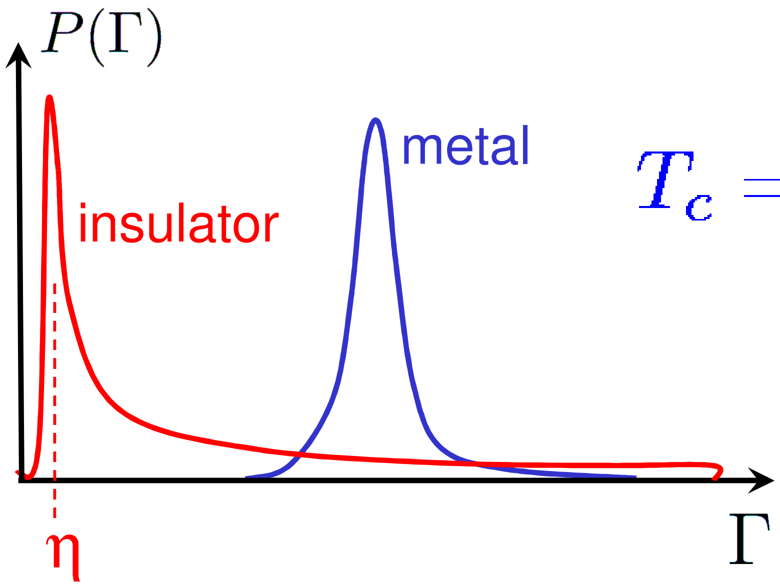
$$\Gamma = \sum_{n,m} \Gamma^{n,m}$$

$$P(\Gamma^{n,m}) = \sqrt{\frac{\gamma^{n,m}}{\pi \left[\Gamma^{n,m}\right]^3}} \exp\left(-\frac{\gamma^{n,m}}{\Gamma^{n,m}}\right)$$

Recall:

$$\gamma^{n,m} \leq \eta \left(\frac{T}{T_c}\right)^n$$

$$T_c = \frac{\delta_\zeta}{12\lambda M |\ln\lambda|}\left[1 + \mathcal{O}\left(\lambda M \ln I\right)\right]$$

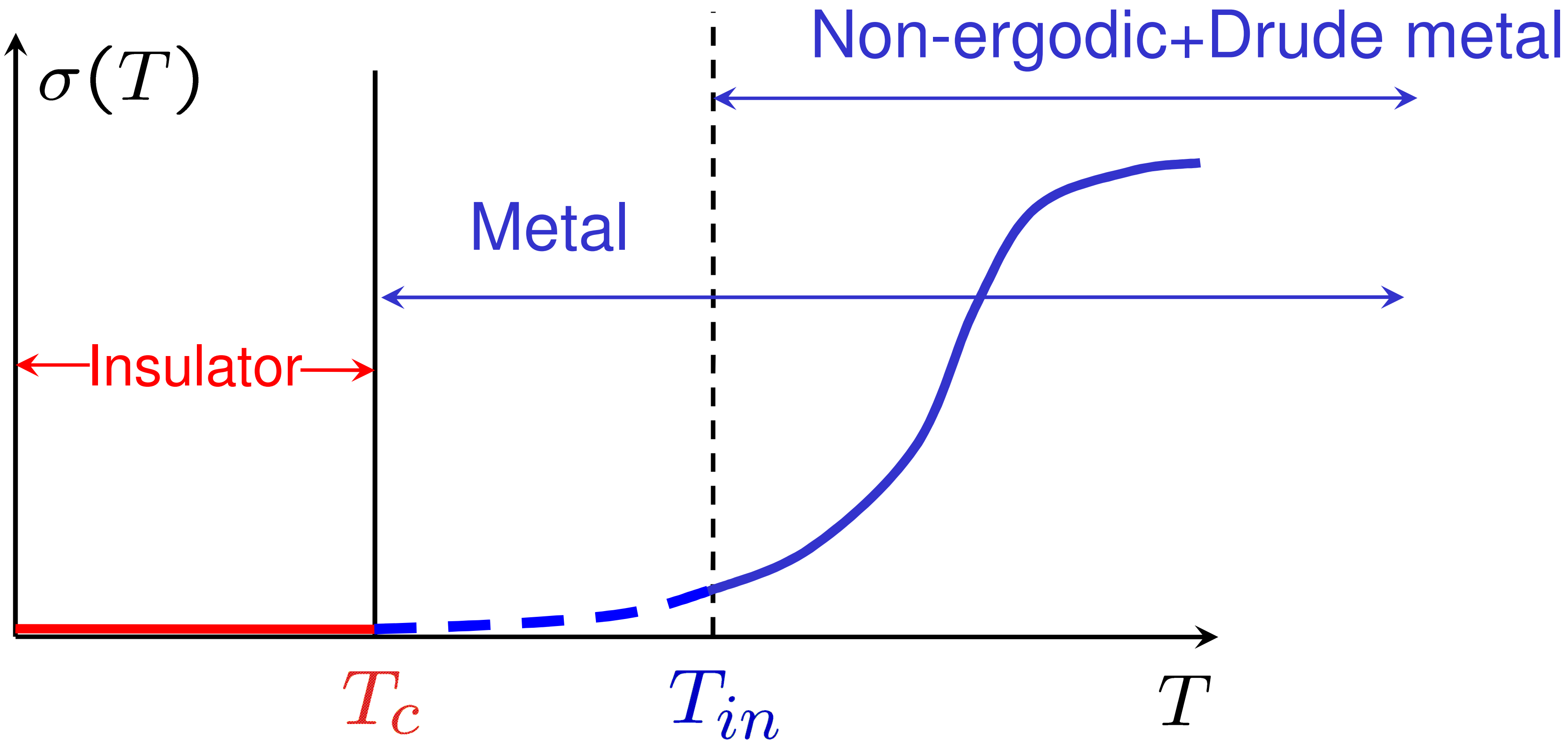


probability distribution
for a fixed energy

$T < T_c$ **STABLE**

$T > T_c$ **unstable**

So, we have just learned:



$$T_c = \frac{\delta_\zeta}{12\lambda M |\ln \lambda|}$$

$$T_{in} = \frac{\delta_\zeta}{6\pi\lambda M}$$

Extension to non-degenerate system

$$T_c \gg \epsilon_F$$

$$\hat{H}_{int} = \frac{b}{4} \int d^d \boldsymbol{r} : \left(\hat{\psi}^\dagger \hat{\psi} \right)^2 :, \quad \text{bosons}$$

$$T_c \simeq \frac{\delta_\zeta^2(T_c)}{b n_0}; \quad \text{if} \quad \frac{d\zeta(\epsilon)}{d\epsilon} > 0$$

For 1D it leads to:

$$\frac{\hbar^2}{m\zeta(T_c)^2} \simeq b n_0;$$

I.A. and B.L. Altshuler , unpublished (2008)

Estimate for the transition temperature for general case

1) Start with T=0;

2) Identify elementary (one particle) excitations and prove that they are localized.

3) Consider a one particle excitation at finite T and the possible paths of its decays:

$$¢_t(T_c)\ '\ U(T_c)N_1(T_c)$$

Energy mismatch

Interaction matrix element

# of possible decay processes of an excitations allowed by interaction Hamiltonian;

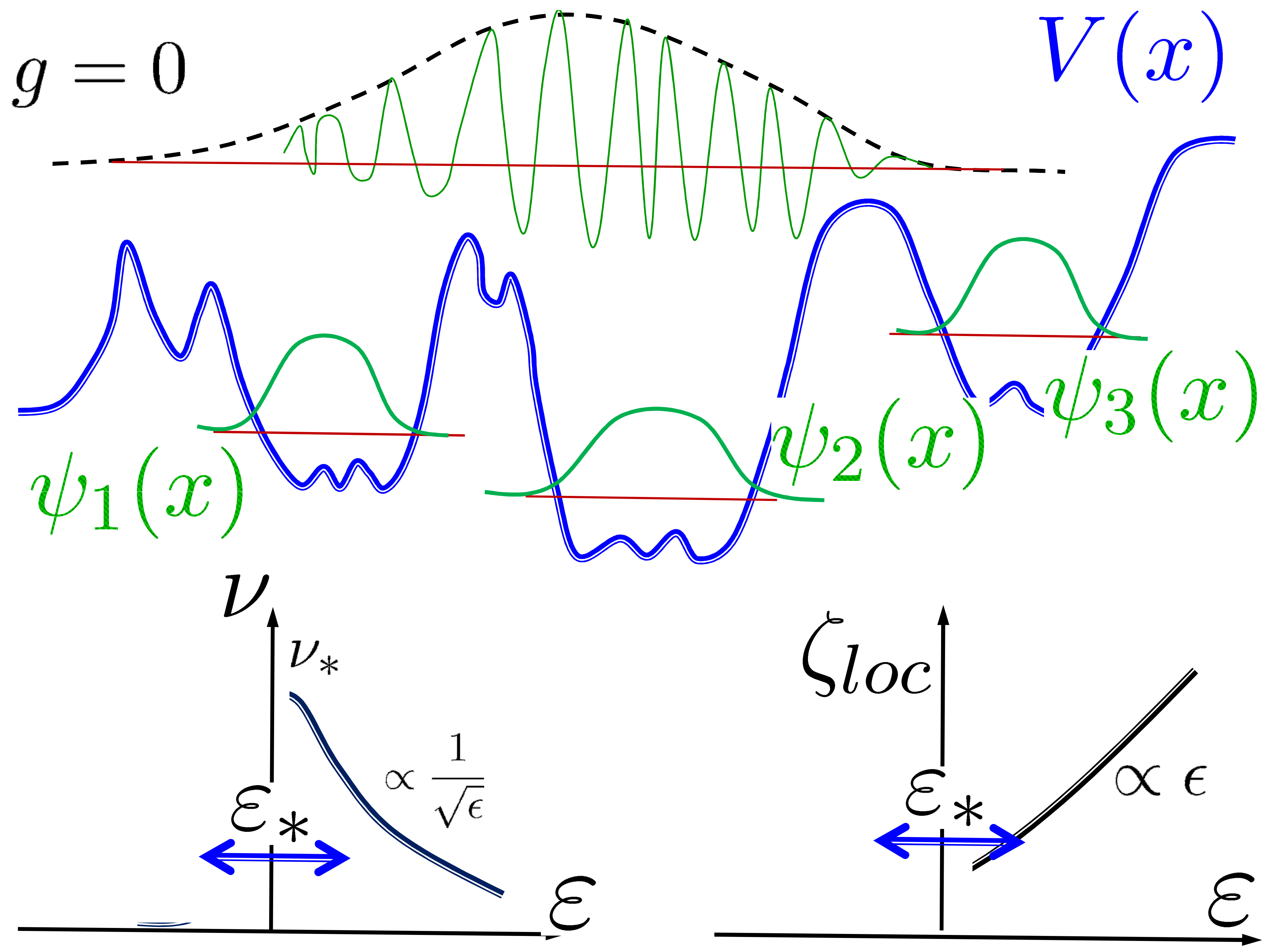
Weakly interacting bosons in one dimension

$$\hat{H} = \int_0^L dx \left[\hat{\psi}^\dagger \left(-\frac{\hbar^2 \partial_x^2}{2m} + V(x) \right) \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right],$$

$$n = \frac{1}{L} \int_0^L dx \hat{\psi}^\dagger(x) \hat{\psi}(x)$$

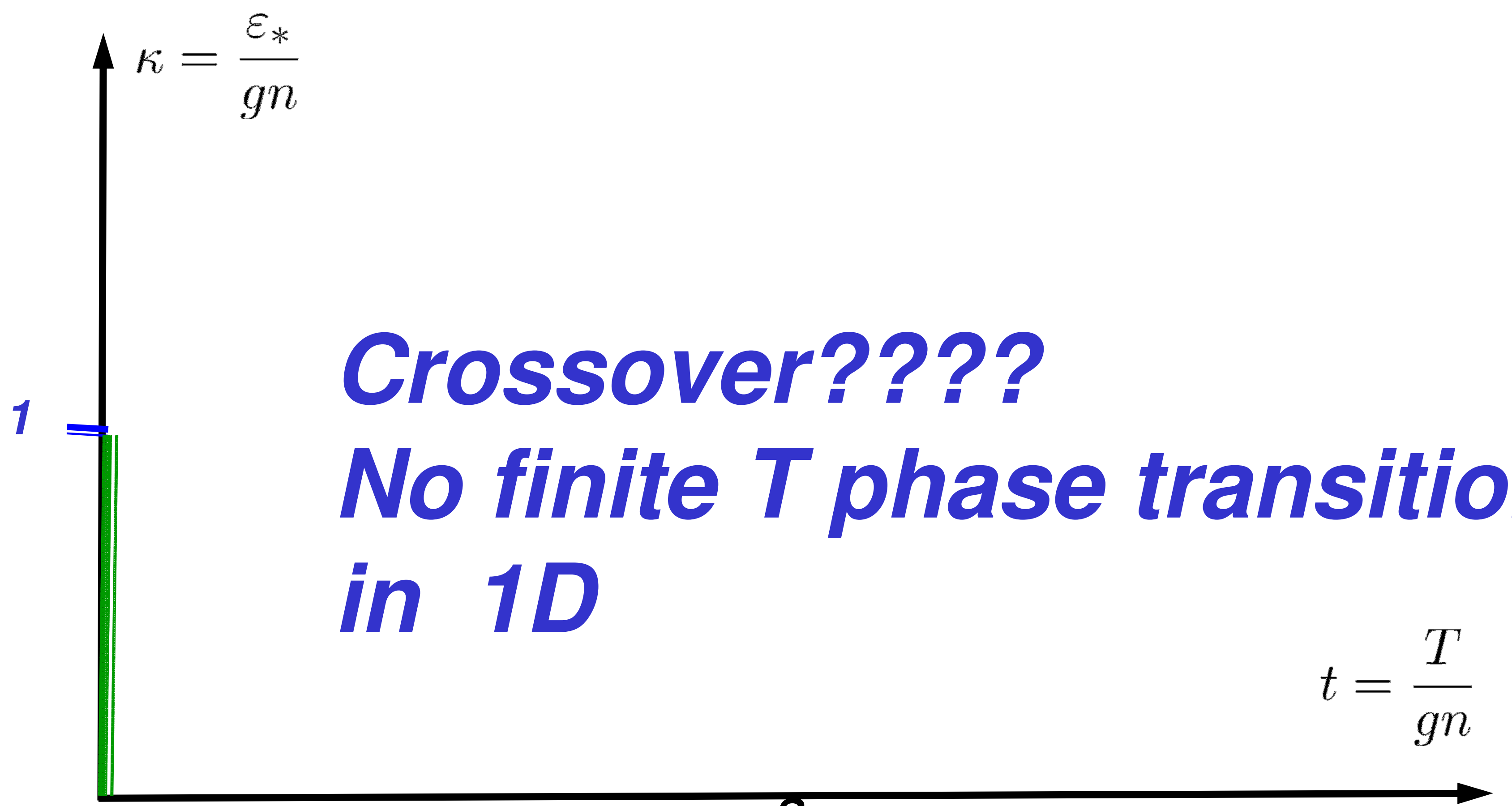
$$0 < \gamma = \frac{gm}{n} \ll 1; \quad L \to \infty$$

Details: Seminar #2
December 28



$g = 0$
$V(x)$
$\psi_1(x)$
$\psi_2(x)$
$\psi_3(x)$
ν
ν_*
$\propto \frac{1}{\sqrt{\epsilon}}$
ε_*
ε
ζ_{loc}
$\propto \epsilon$
ε_*
ε

Phase diagram



$T = 0$ $\quad \kappa < 1; \quad \text{superfluid}$

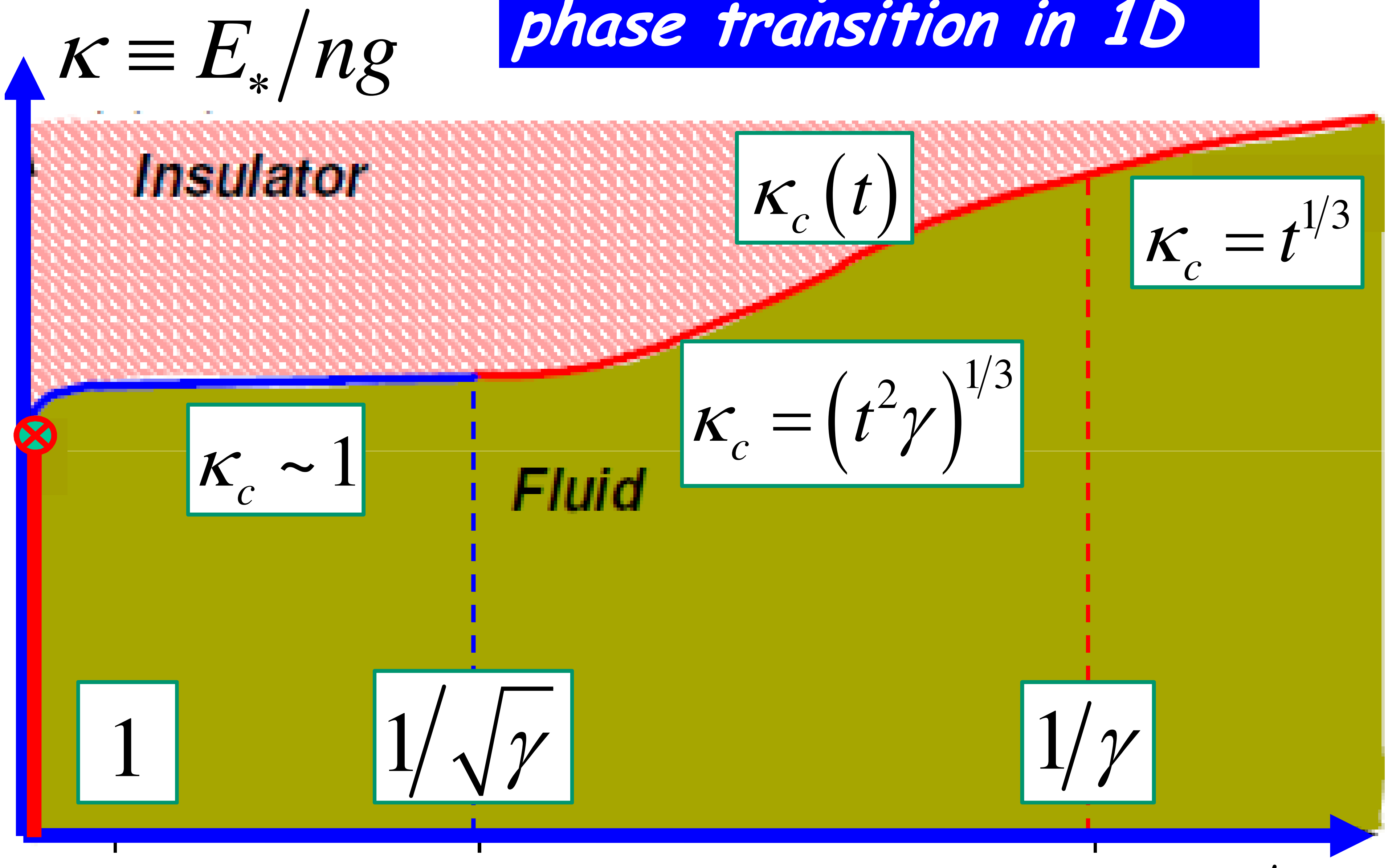
$\qquad\quad \kappa > 1; \quad \text{insulator}$

See e.g.

Altman, Kafri, Polkovnikov, G.Refael, PRL, 100, 170402 (2008); 93,150402 (2004).

G.M. Falco, T. Nattermann, & V.L. Pokrovsky, PRB,80, 104515 (2009) ????.

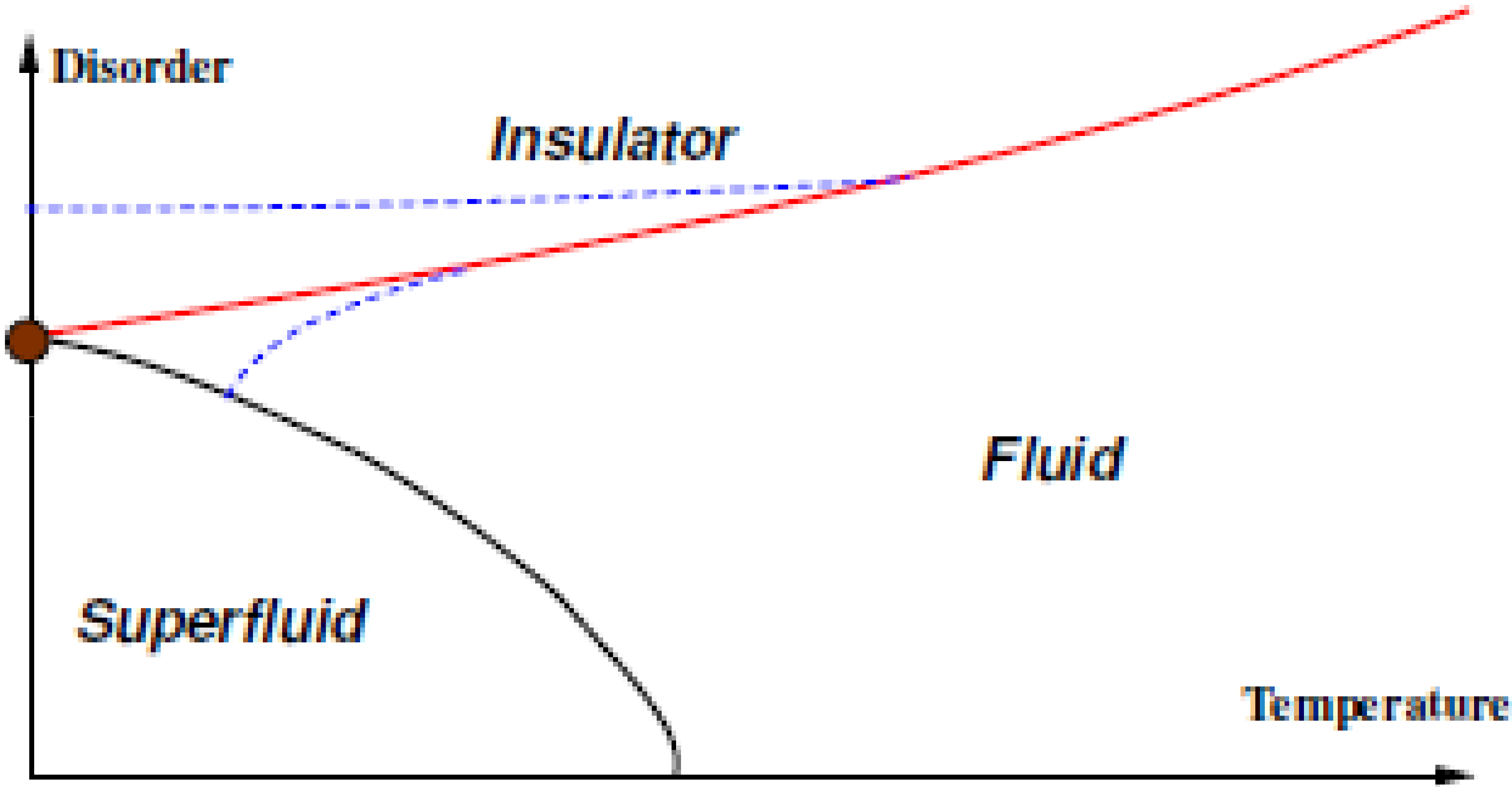
Finite temperature phase transition in 1D



$\gamma = \frac{gm}{n} \ll 1$

I.A., Altshuler, Shlyapnikov
arXiv:0910.434; Nature Physics (2010)

Disordered interacting bosons in two dimensions



Conclusions:

- Existence of the many-body mobility threshold is established.
- The many body metal-insulator transition is ***not*** a thermodynamic phase transition.
- It is associated with the vanishing of the Langevine forces rather the divergences in energy landscape (like in classical glass)
- Only phase transition possible in one dimension (for local Hamiltonians)