Many body localization of weakly interacting disordered fermions

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Detailed paper (fermions): Annals of Physics 321 (2006) 1126-1205
Shorter version: cond-mat/0602510; chapter in “Problems of CMP”

Lewiner Institute of Theoretical Physics, Seminar, December 26th, 2010

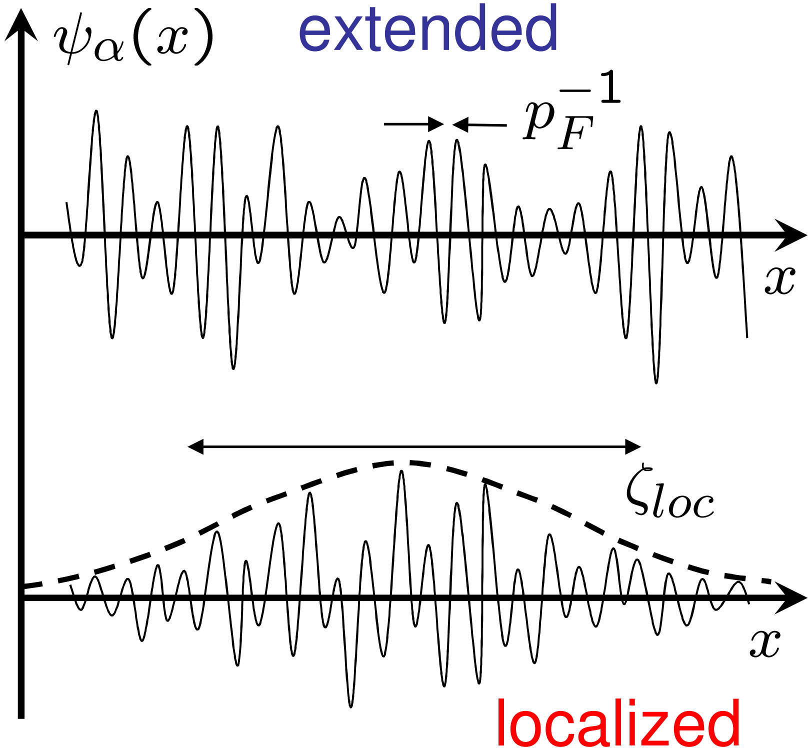
Outline:

- Remind: many body localization
- Effective model for the fermionic systems
- Technique
- Stability of the metal
- Stability of the many-body insulator

- Metal insulator transition

1. Localization of single-electron wave-functions:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha\psi_\alpha(\boldsymbol{r})$$

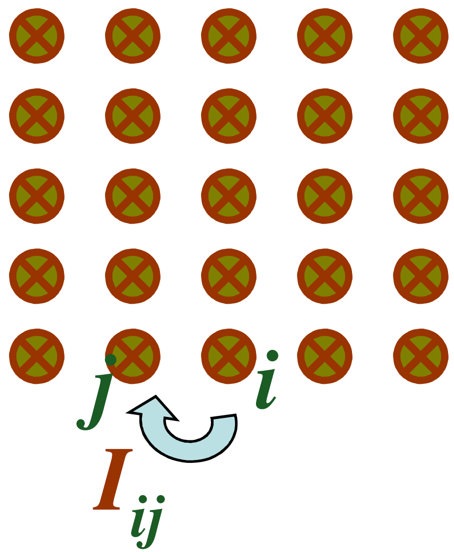


d=1; All states are ***localized***

d=2; All states are ***localized***

d>2; ***Anderson transition***

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

$$-W < \varepsilon_i < W$$

uniformly distributed

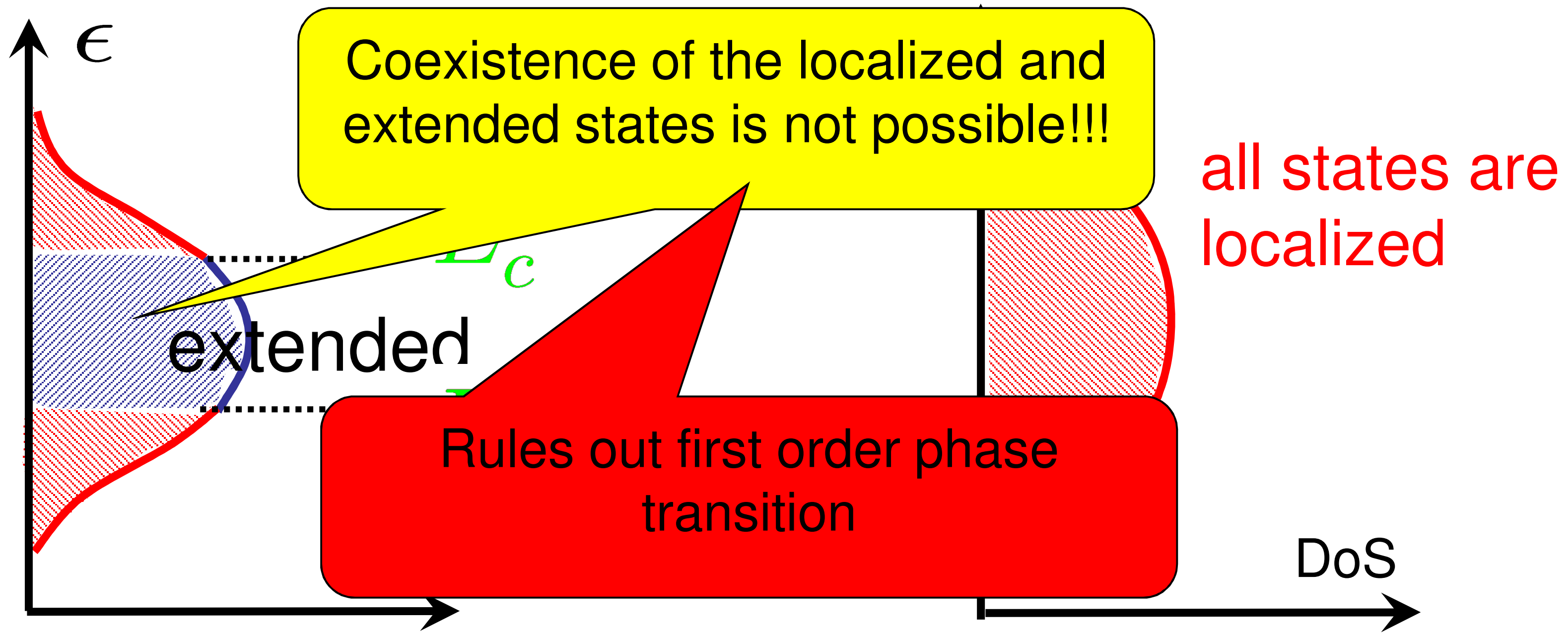
Critical hopping:

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)$$

$$d \gtrsim 3 \gg 1$$

Anderson Transition

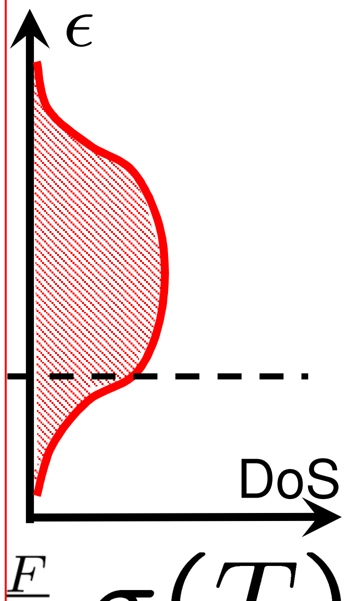
$I > I_c$ $\qquad\qquad$ $I < I_c$



E_c - mobility edges (one particle)

<u>*Temperature dependence of the conductivity (I)*</u>

Assume that all the states are <u>*localized*</u>



$$\sigma(T) = 0$$

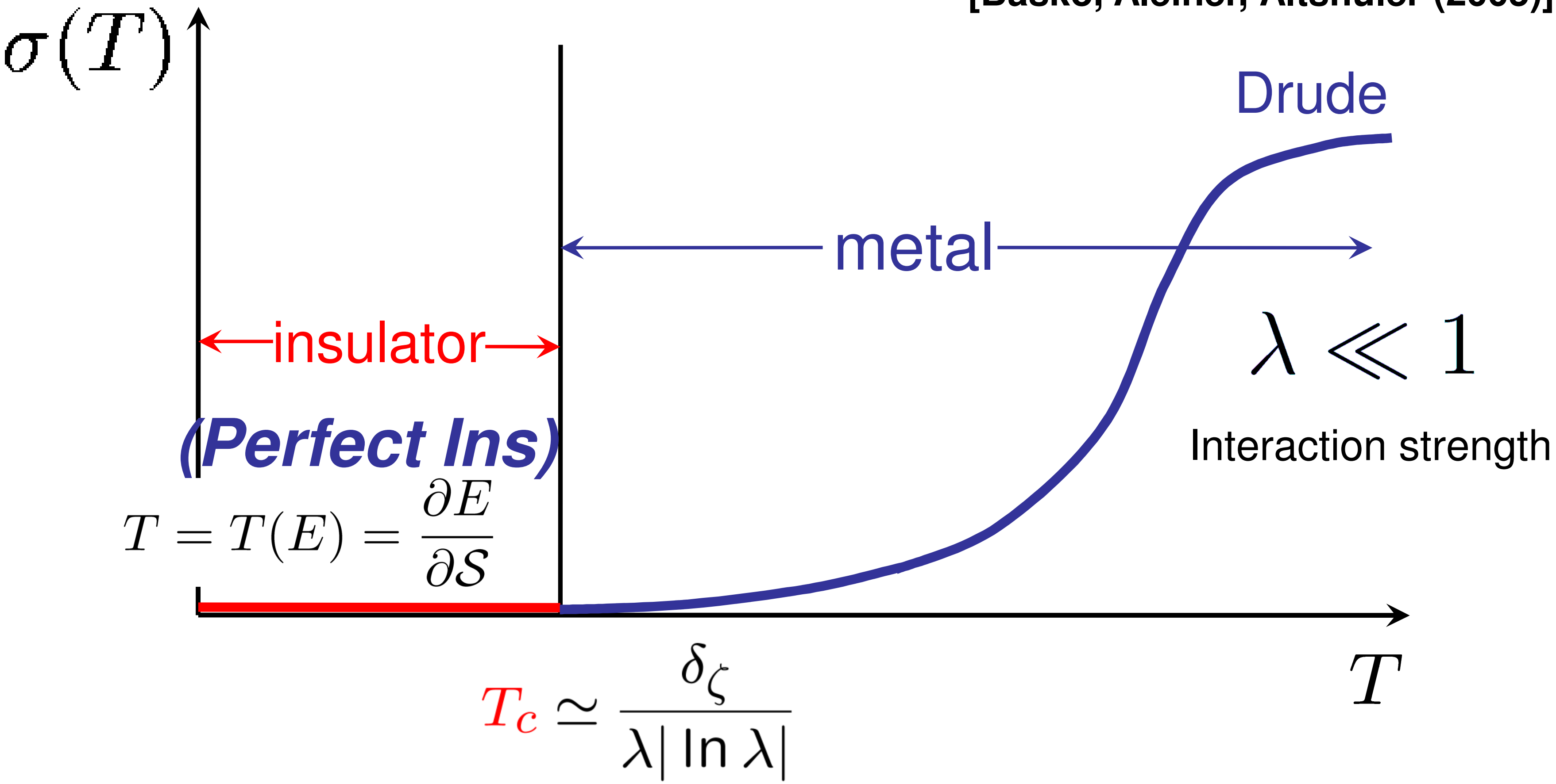
Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

A#1: Sure ***[Person from the street (2005)]***

A#2: No way [L. Fleishman. P.W. Anderson (1980)]

A#3: Finite T *Metal-Insulator Transition*

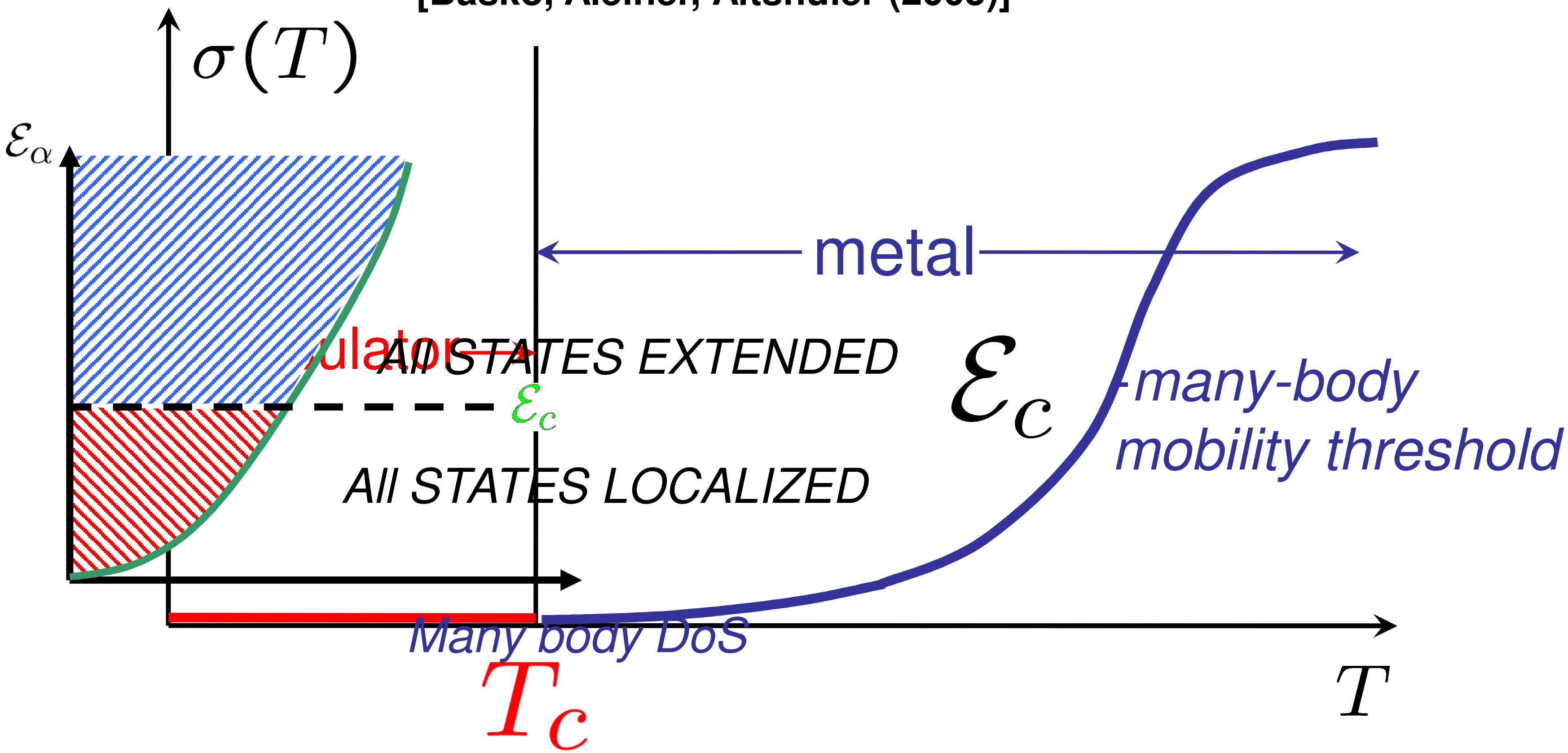
[Basko, Aleiner, Altshuler (2005)]



Many-body mobility threshold

$$\left[\hat{H}_1 + \hat{H}_{int}\right]\Psi_\alpha = \mathcal{E}_\alpha \Psi_\alpha$$

[Basko, Aleiner, Altshuler (2005)]



"All states are *localized* "

means

Probability to find an extended state:

$$\mathcal{P}_{ext} \propto \exp\left(-\#\frac{\mathcal{V}}{\mathcal{V}_{loc}(\mathcal{E})}\right)$$

$$\lim_{\mathcal{E}\to\mathcal{E}_c-0} \mathcal{V}_{loc}(\mathcal{E}) = \infty$$

System volume

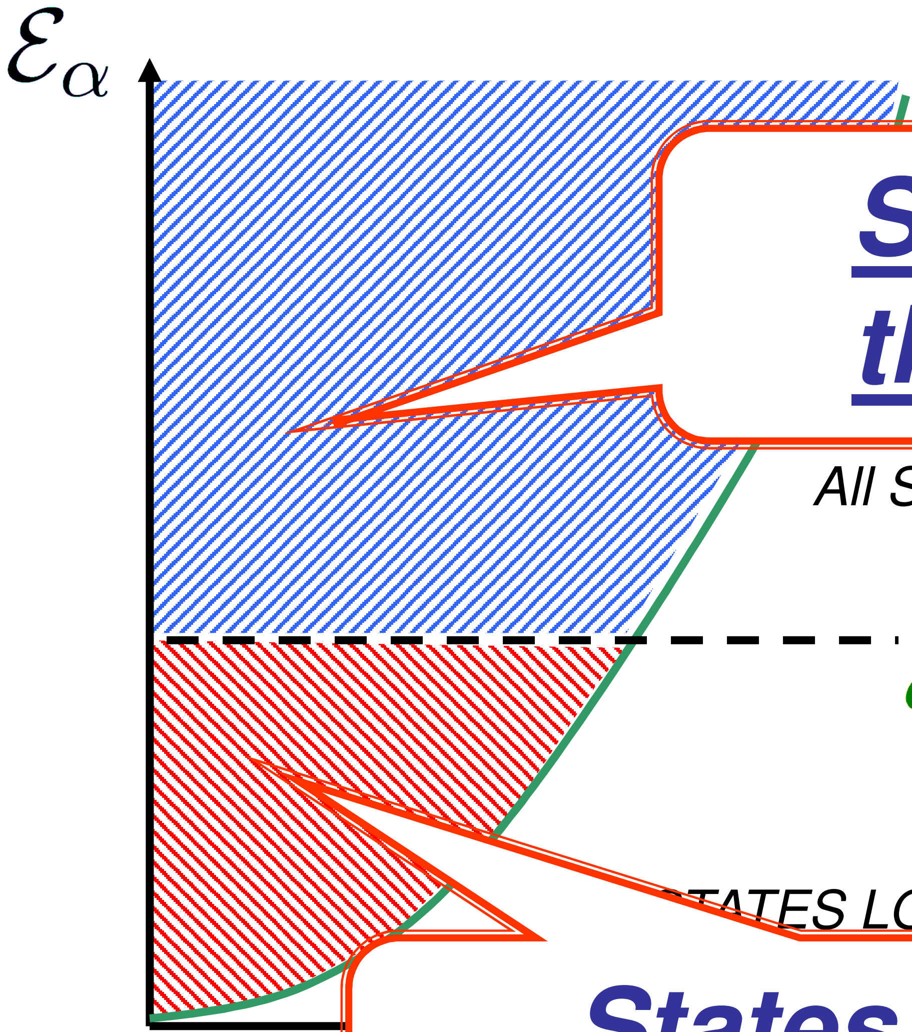
Localized one-body wave-function

Means, in particular:

$$\langle i\,|O(\boldsymbol{r}_1)|\,j\rangle\langle j\,|O(\boldsymbol{r}_2)|\,i\rangle \simeq \begin{cases} a\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{L(\omega)}\right), & \omega=\xi_i-\xi_j \quad \textit{extended} \\ b\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\zeta_{loc}}\right), & \textit{localized} \end{cases}$$

We define localized many-body wave-function as:

$$\langle\alpha\left|\hat{O}(\boldsymbol{r}_1)\right|\beta\rangle\langle\beta\left|\hat{O}(\boldsymbol{r}_2)\right|\alpha\rangle \simeq \begin{cases} \mathcal{A}\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{L(\omega)}\right), & \omega=\mathcal{E}_\alpha-\mathcal{E}_\beta \quad \textit{extended} \\ \mathcal{B}\left(\dfrac{|\boldsymbol{r}_1-\boldsymbol{r}_2|}{\zeta_{loc}}\right), & \textit{localized} \end{cases}$$



States always thermalized!!!

All STATES EXTENDED

$$\mathcal{E}_c = \int_0^{T_c} C_V(T)dT$$

TATES LOCALIZED

States never thermalized!!!

Entropy

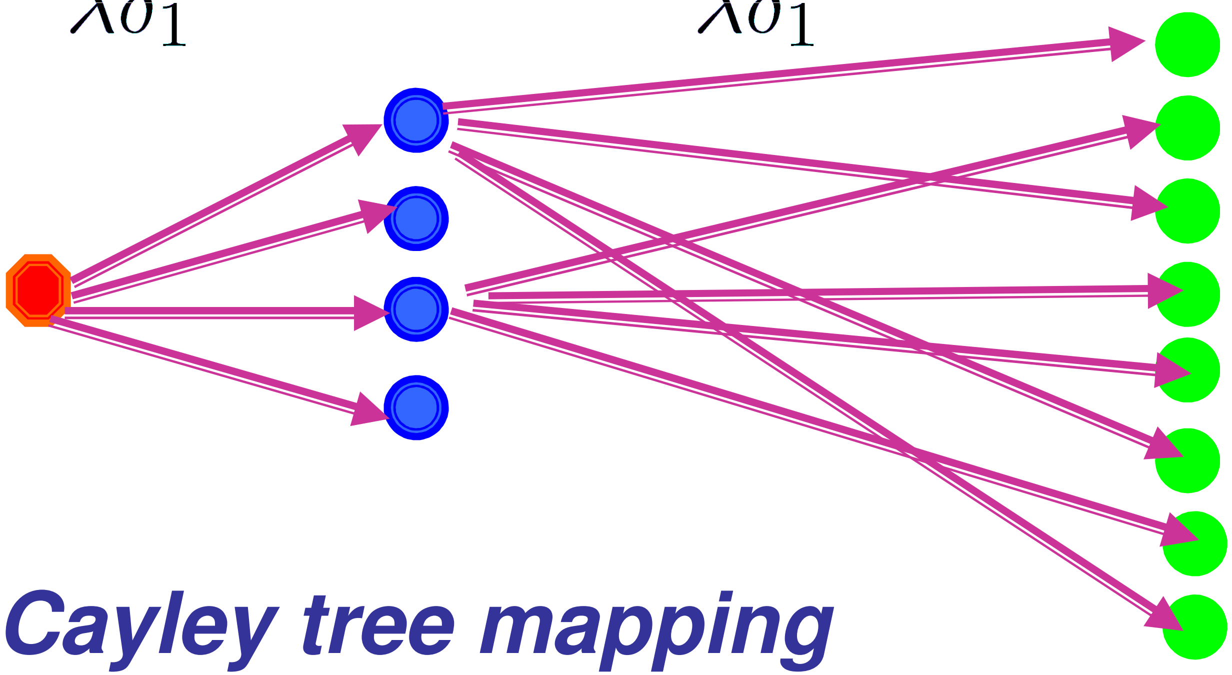
$\propto \exp[S(\mathcal{E})]$

Fock space localization in quantum dots (AGKL, 1997)

$$\hat{H} = \sum_{\alpha} \xi_\alpha \hat{c}^\dagger_\alpha \hat{c}_\alpha + \ldots + \lambda\delta_1 \sum_{\alpha\beta\gamma\delta} (\pm) \hat{c}^\dagger_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma \hat{c}_\delta$$

***1*-particle excitation** ***3*-particle excitation** ***5*-particle excitation**

$$\xi_\alpha \quad \xrightarrow[\lambda\delta_1]{} \quad \xi_\gamma + \xi_\delta - \xi_\beta \quad \xrightarrow[\lambda\delta_1]{} \quad \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \ldots \quad \lambda\delta_1$$



Cayley tree mapping

<u>Effective</u> Hamiltonian for ***<u>MIT</u>***.

We would like to describe the low-temperature regime only.

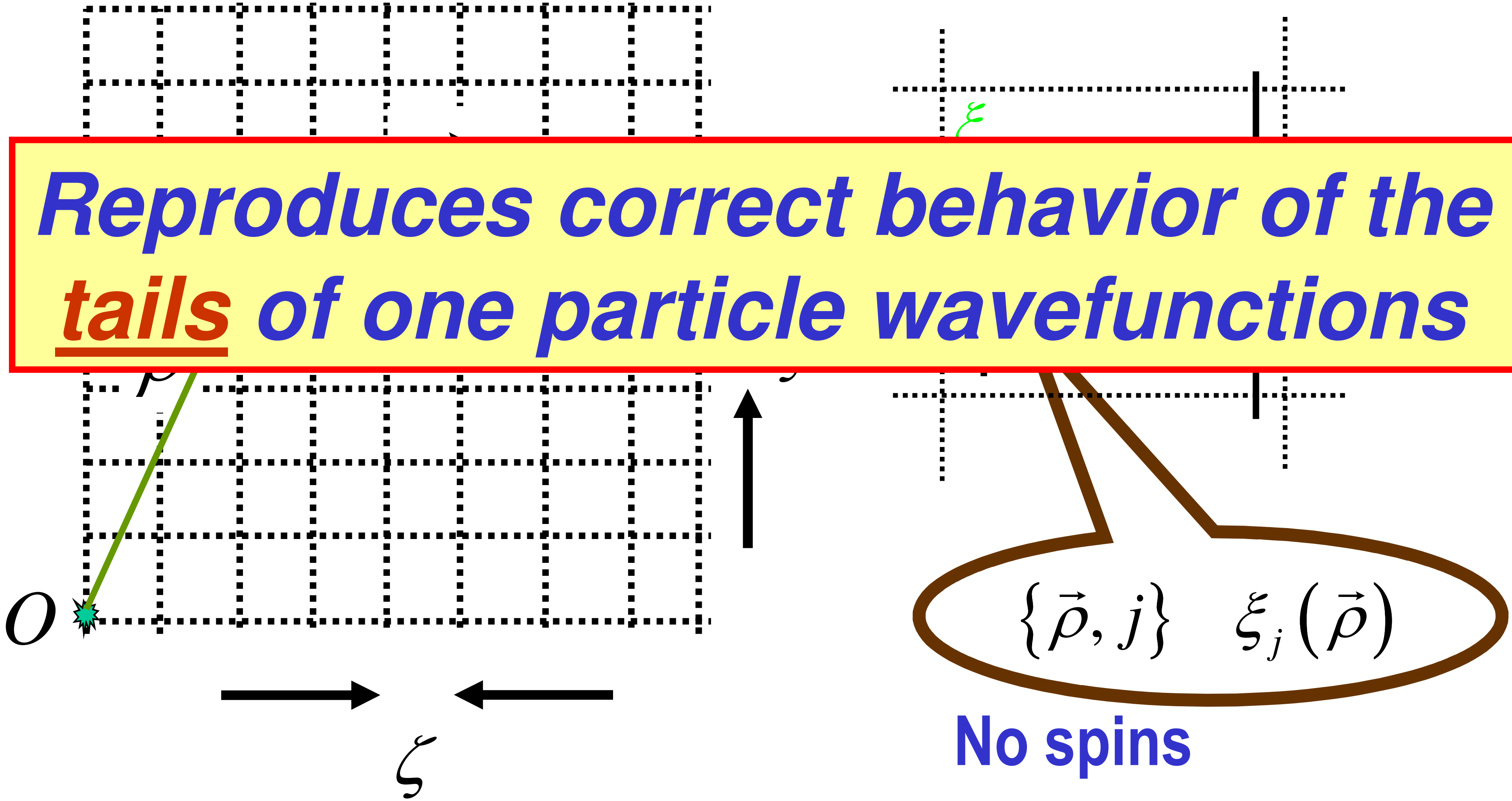
Spatial scales of interest >> ζ_{loc}

1-particle localization length

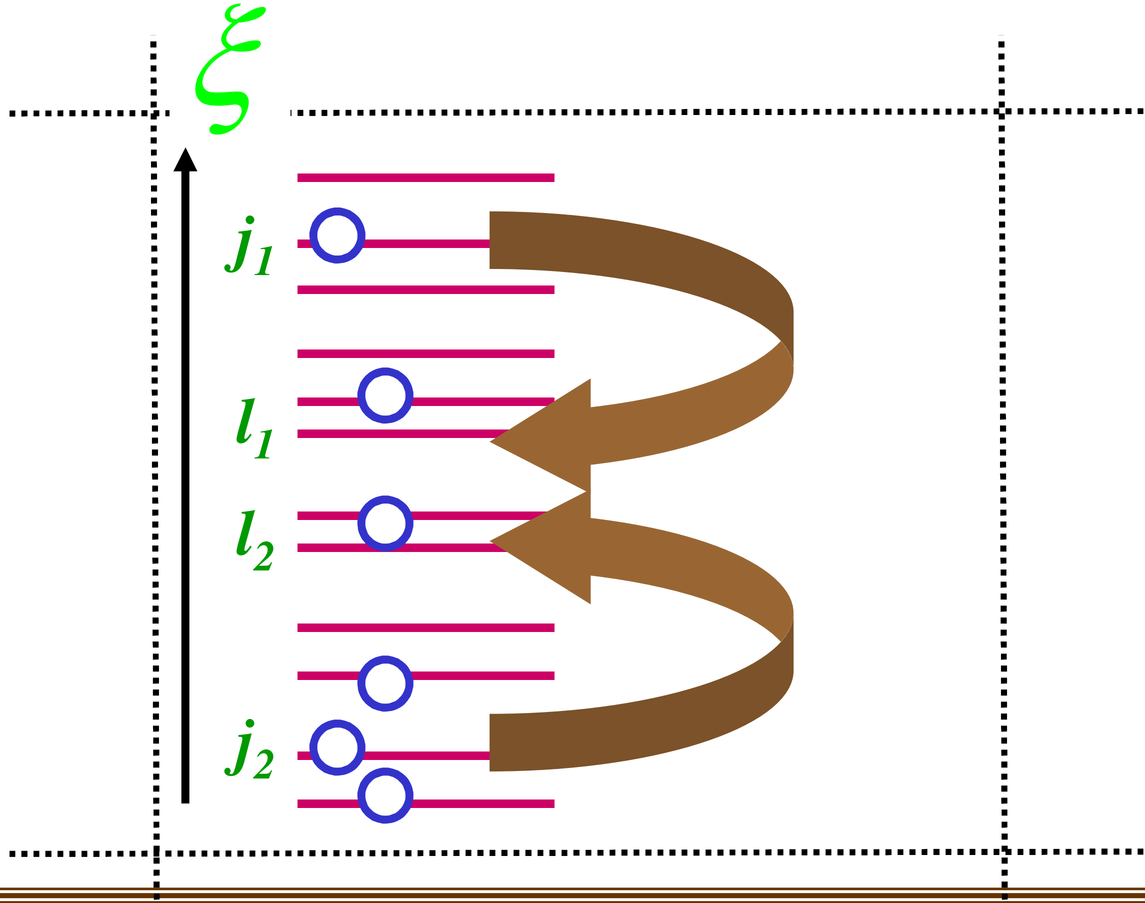
Otherwise, conventional perturbation theory for disordered metals works.

Altshuler, Aronov, Lee (1979); Finkelshtein (1983) – T-dependent SC potential
Altshuler, Aronov, Khmelnitskii (1982) – inelastic processes

Reproduces correct behavior of the ***tails*** *of one particle wavefunctions*



$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho} + \boldsymbol{a}) \right]$$



$$\hat{V}_{int} = \frac{1}{2} \sum_{l_1 l_2 j_1 j_2;\boldsymbol{\rho}} V_{l_1 l_2}^{j_1 j_2}(\boldsymbol{\rho}) \hat{c}_{l_1}^{\dagger}(\boldsymbol{\rho}) \hat{c}_{l_2}^{\dagger}(\boldsymbol{\rho}) \hat{c}_{j_2}(\boldsymbol{\rho}) \hat{c}_{j_1}(\boldsymbol{\rho})$$

Interaction only within the same cell;

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

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Statistics of matrix elements?

Energy transfer $\omega \gg \delta_\zeta$

corresponds to the special scale $L_\omega = \sqrt{D/\omega} \ll \zeta$.

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \hat{c}_l^\dagger(\boldsymbol{\rho}) \left[\xi_l(\boldsymbol{\rho})\hat{c}_l(\boldsymbol{\rho}) + I\delta_\xi \sum_{\boldsymbol{a},m} \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

$$\hat{V}_{int} = \frac{1}{2} \sum_{l_1 l_2 j_1 j_2;\boldsymbol{\rho}} V_{l_1 l_2}^{j_1 j_2}(\boldsymbol{\rho}) \hat{c}_{l_1}^\dagger(\boldsymbol{\rho}) \hat{c}_{l_2}^\dagger(\boldsymbol{\rho}) \hat{c}_{j_2}(\boldsymbol{\rho}) \hat{c}_{j_1}(\boldsymbol{\rho})$$

$$V_{l_1 l_2}^{j_1 j_2} = \frac{\lambda\delta_\zeta \sigma_{l_1}^{j_1} \sigma_{l_2}^{j_2}}{2} \Upsilon\left(\frac{\xi_{j_1}-\xi_{l_1}}{\delta_\zeta}\right) \Upsilon\left(\frac{\xi_{j_2}-\xi_{l_2}}{\delta_\zeta}\right) - (l_1 \leftrightarrow l_2)$$

$$\Upsilon(x) = \theta\left(\frac{M}{2} - |x|\right); \quad 1 \ll M \lesssim \frac{1}{\sqrt{\lambda}}$$

Parameters: $\lambda, I, M^{-1} \ll 1$

σ_l^j **random signs**

$$\hat{H}_0 = \sum_{\boldsymbol{\rho},l} \left[\xi_l(\boldsymbol{\rho}) \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_l(\boldsymbol{\rho}) + I\delta_\zeta \sum_{\boldsymbol{a},m} \hat{c}_l^\dagger(\boldsymbol{\rho}) \hat{c}_m(\boldsymbol{\rho}+\boldsymbol{a}) \right]$$

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σ_l^j **random signs**

Ensemble averaging over: $\xi_l(\rho); \sigma_i^j = \pm 1$

Level repulsion: Only within one cell.

Probability to find n levels in the energy interval of the width E:

$$P(n,E) = \frac{e^{-E/\delta_\zeta}}{n!} \left(\frac{E}{\delta_\zeta}\right)^n \exp\left[-F\left(\frac{n\delta_\zeta}{E}\right)\right]$$

$$\lim_{x\to\infty} \frac{F(x)}{x} = \infty$$

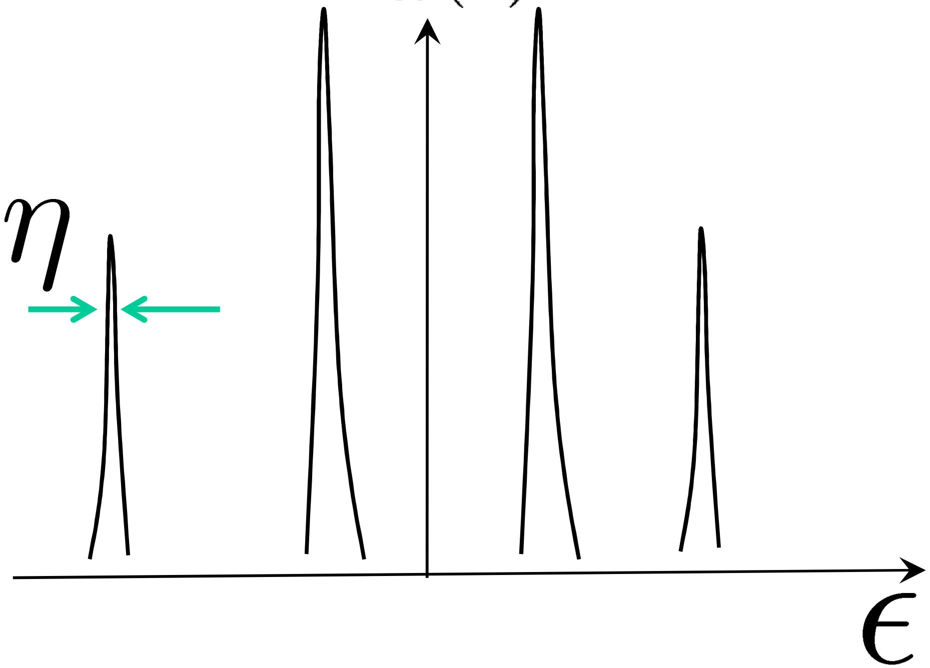
What to calculate?

Idea for one particle localization Anderson, (1958);
MIT for Cayley tree: Abou-Chakra, Anderson, Thouless (1973);
Critical behavior: Efetov (1987)

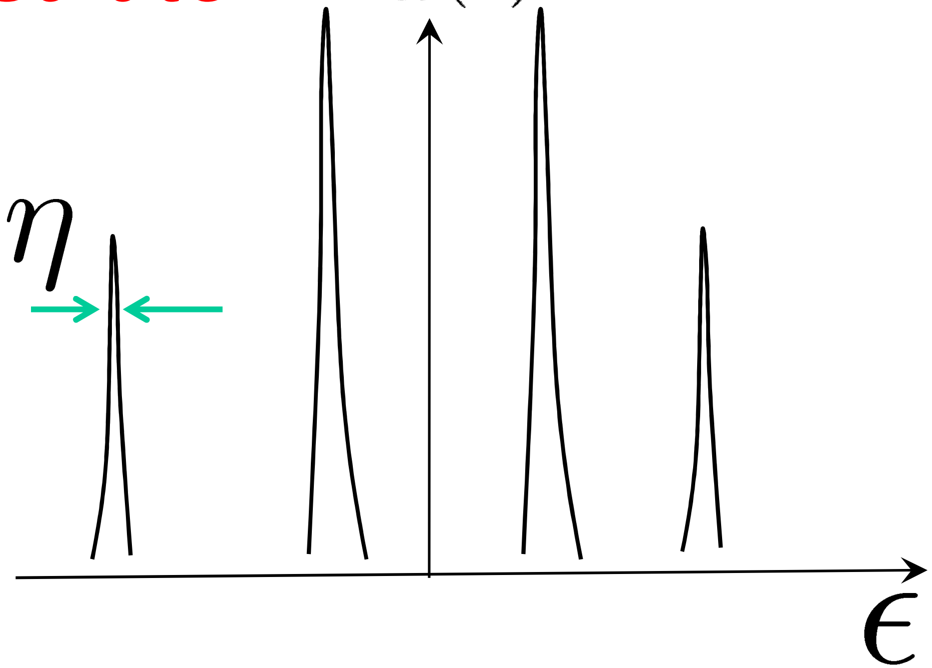
$\Gamma_\alpha(\epsilon) = \mathrm{Im}\, \Sigma_\alpha^A(\epsilon)$ – random quantity

No interaction: $\Gamma_\alpha(\epsilon) = \eta \rightarrow +0$

Metal $\Gamma_\alpha(\epsilon)$



Insulator $\Gamma_\alpha(\epsilon)$



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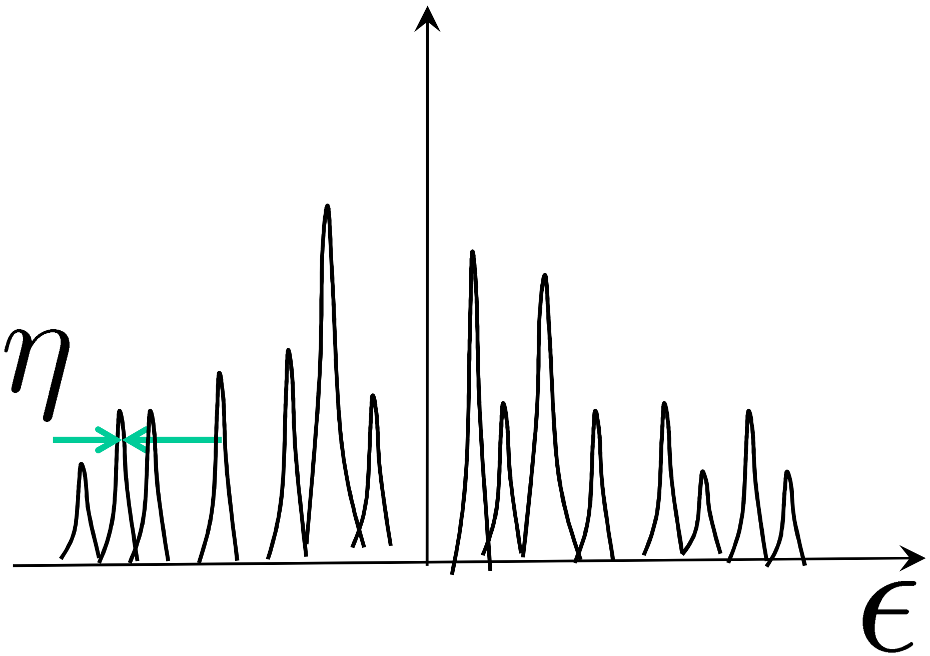

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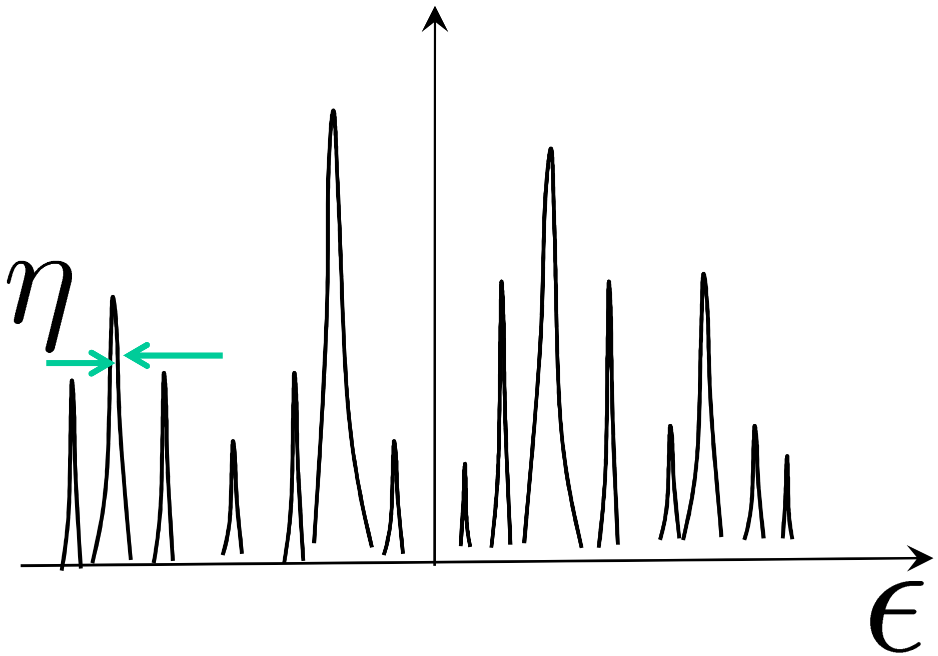

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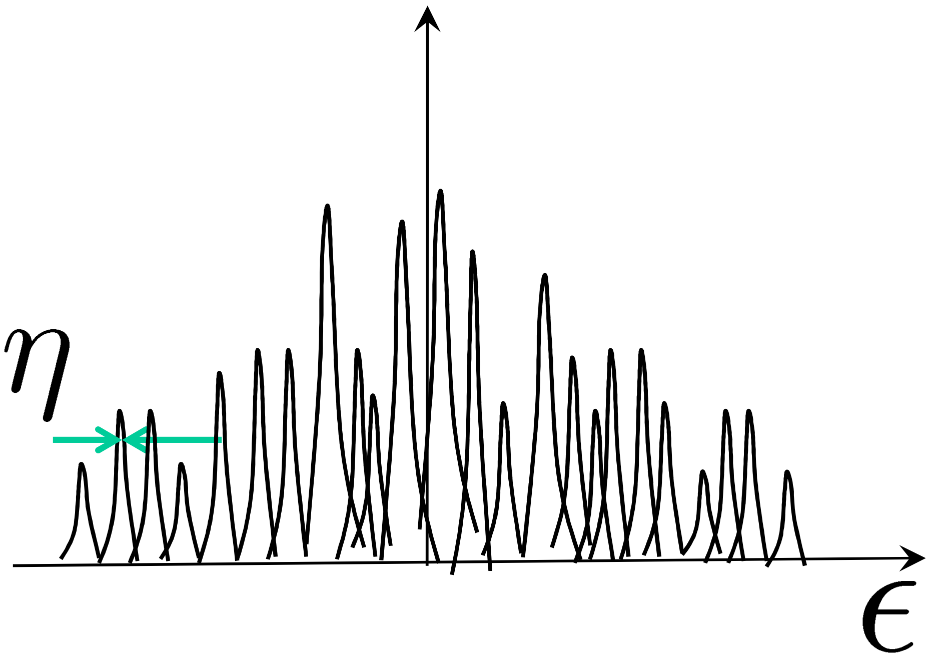

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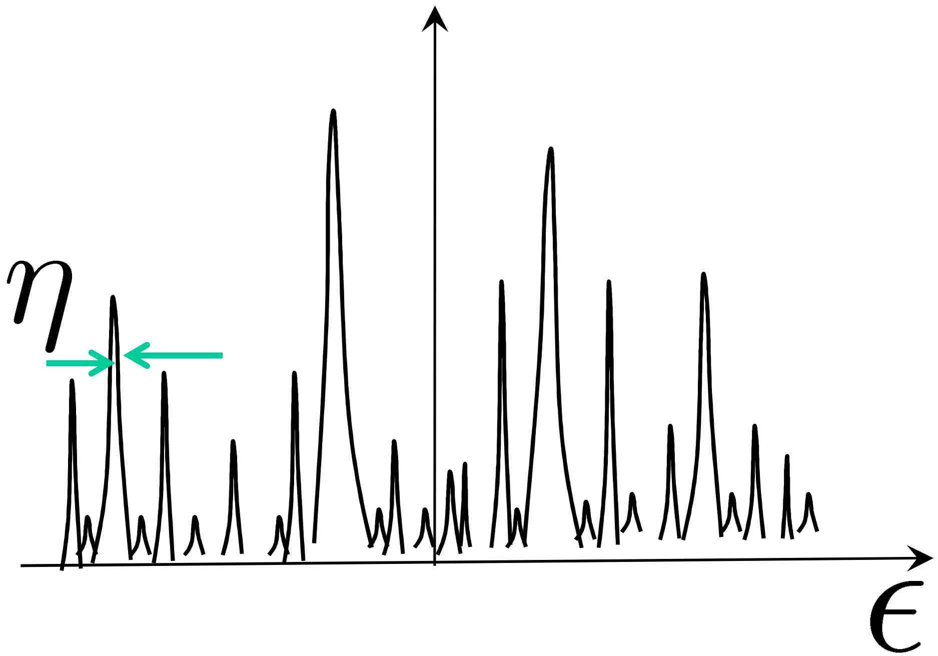

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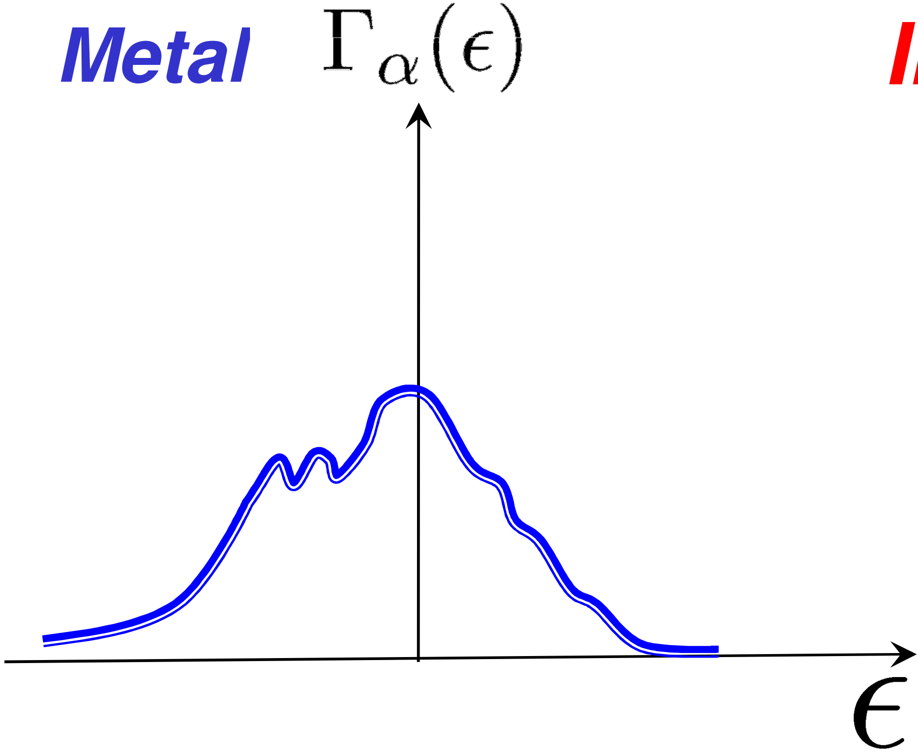

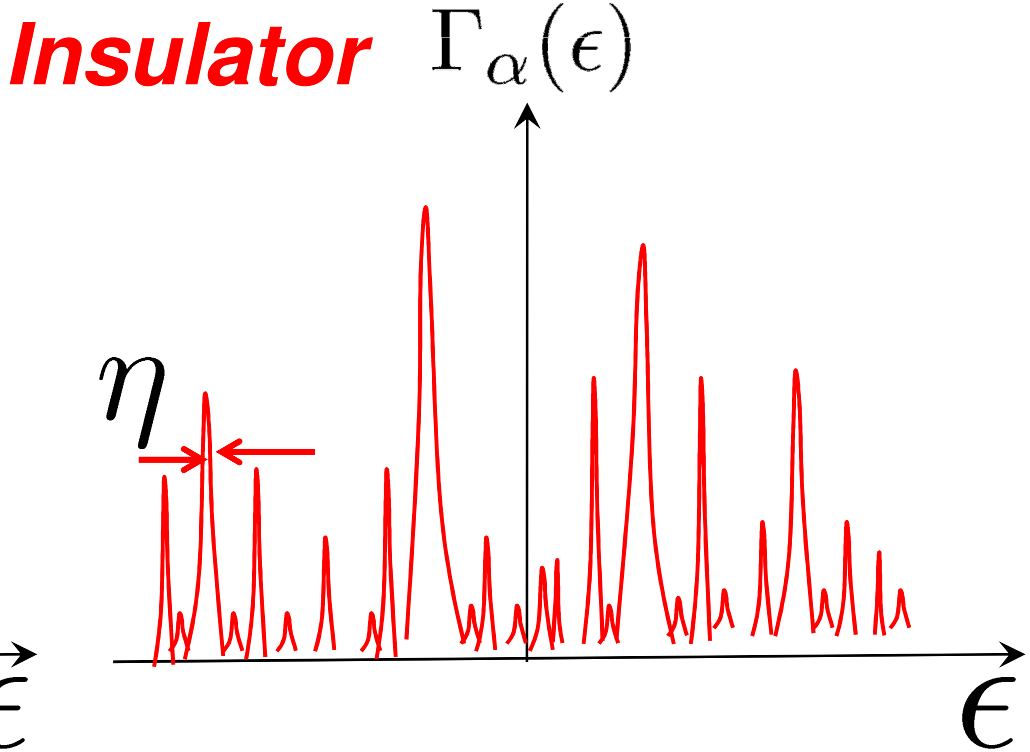

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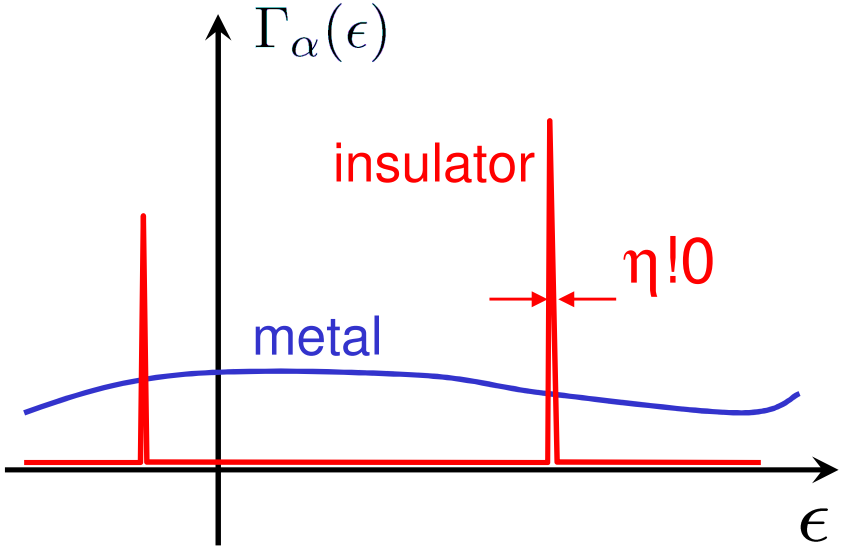
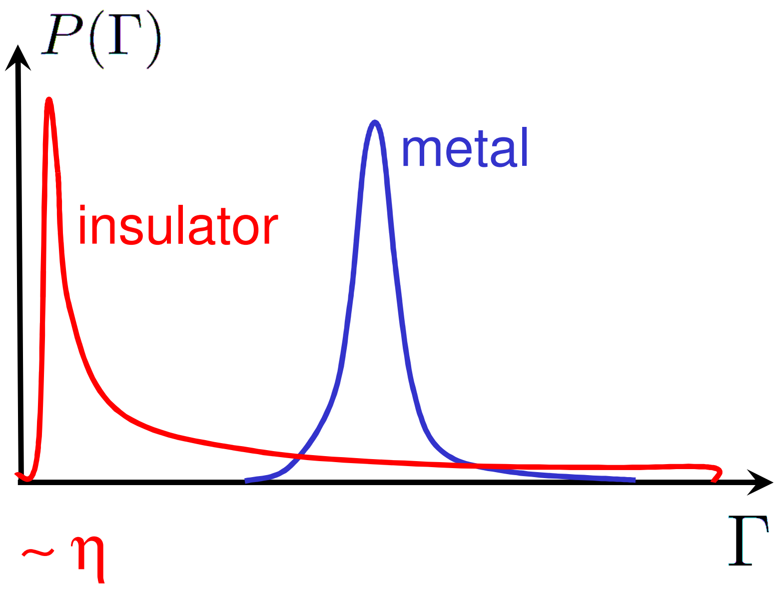
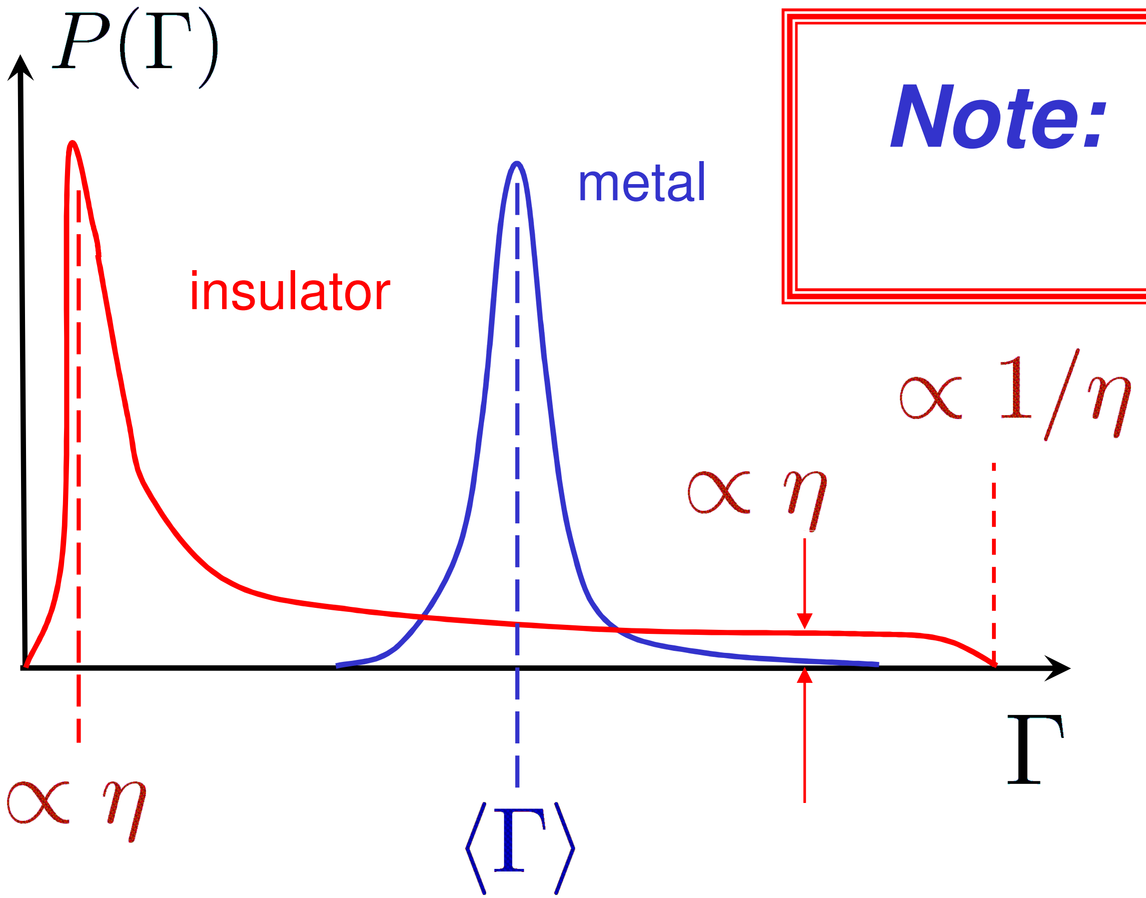


behavior for a
given realization



probability distribution
for a fixed energy

Probability Distribution



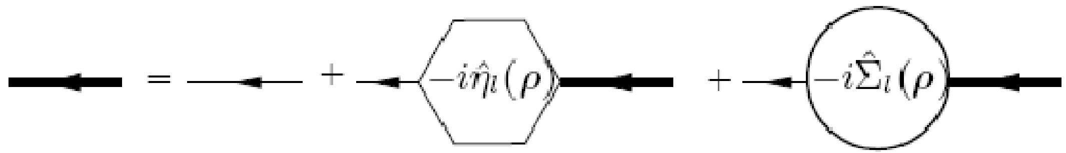
Note: $\langle\Gamma\rangle = \langle\Gamma\rangle$

Look for:

$$\lim_{\eta\to+0}\lim_{\mathcal{V}\to\infty} P(\Gamma>0) = \begin{cases} >0; & metal \\ 0; & insulator \end{cases}$$

How to calculate?

non-equilibrium (arbitrary occupations) → Keldysh

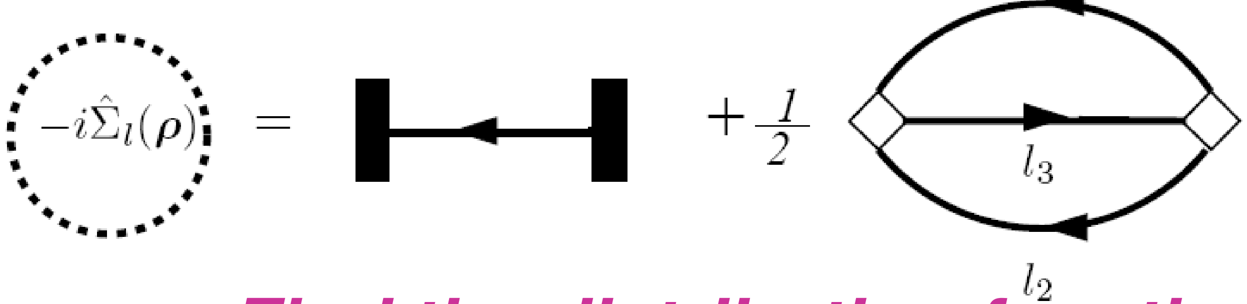


Parameters:

$$\lambda, I, M^{-1} \ll 1$$

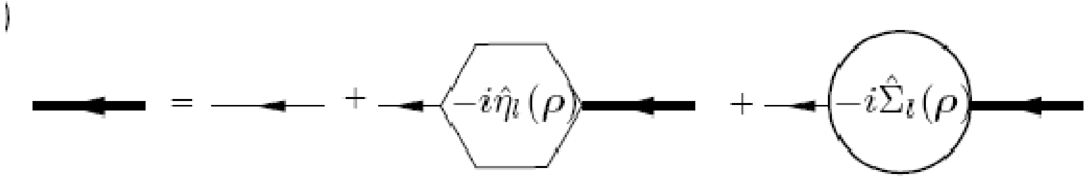
allow to select the most relevant series

(a)

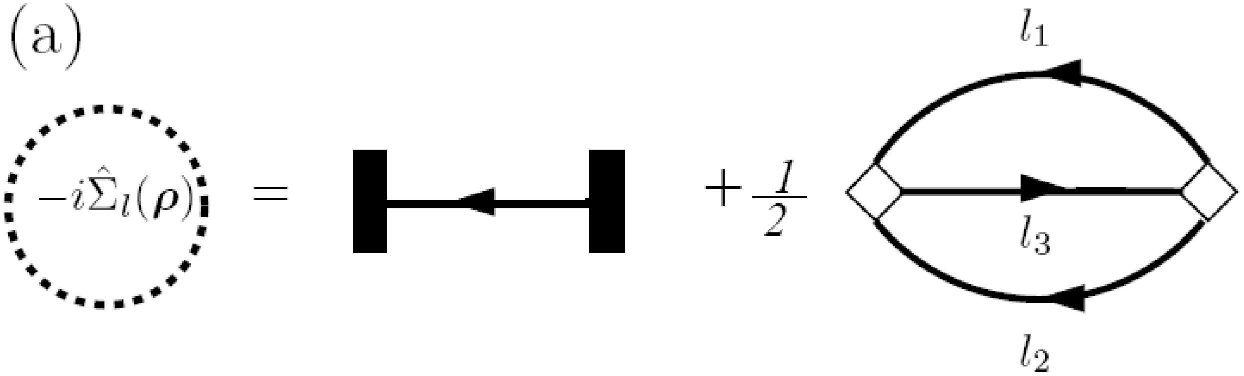


SCBA

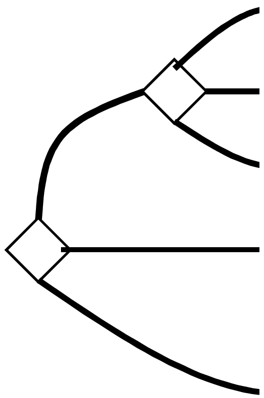
Find the distribution function of each diagram



(a)



Iterations:



Cayley tree structure

Nonlinear integral equation with random coefficients

after standard simple tricks:

Decay due to tunneling

Decay due to e-h pair creation

$$\Gamma_l(\epsilon) = \Gamma_l^{(el)}(\epsilon) + \Gamma_l^{(in)}(\epsilon) + \eta;$$

$$\Gamma_l^{(el)}(\epsilon, \boldsymbol{\rho}) = \pi I^2 \delta_\zeta^2 \sum_{l_1, \boldsymbol{a}} A_{l_1}(\epsilon, \boldsymbol{\rho} + \boldsymbol{a})$$

$$\Gamma_l^{(in)}(\epsilon) = \pi \lambda^2 \delta_\zeta^2 \sum_{l_1,l_2,l_3} Y_{l_1,l_2}^{l_3,l} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 A_{l_1}(\epsilon_1) A_{l_2}(\epsilon_2) A_{l_3}(\epsilon_3) \delta(\epsilon - \epsilon_1 - \epsilon_2 + \epsilon_3) F_{l_1,l_2;l_3}^{\Rightarrow}(\epsilon_1, \epsilon_2; \epsilon_3);$$

$$A_l(\epsilon) = \frac{\pi^{-1}\Gamma_l(\epsilon)}{[\epsilon - \xi_l]^2 + [\Gamma_l(\epsilon)]^2}$$

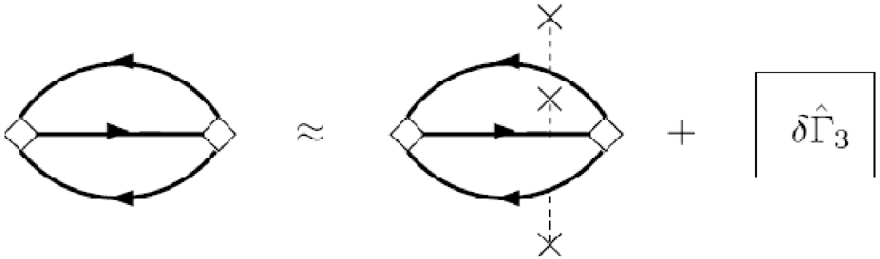
$$Y_{l_1,l_2}^{l_3,l} \equiv \frac{1}{2}\left[\Upsilon\left(\frac{\xi_{l_2} - \xi_l}{\delta_\zeta}\right)\Upsilon\left(\frac{\xi_{l_1} - \xi_{l_3}}{\delta_\zeta}\right) - \Upsilon\left(\frac{\xi_{l_1} - \xi_l}{\delta_\zeta}\right)\Upsilon\left(\frac{\xi_{l_2} - \xi_{l_3}}{\delta_\zeta}\right)\right]^2$$

$$F_{l_1,l_2;l_3}^{\Rightarrow}(\epsilon_1, \epsilon_2; \epsilon_3) = \frac{1}{4}\left\{1 + n_{l_1}(\epsilon_1) n_{l_2}(\epsilon_2) - n_{l_3}(\epsilon_3)\left[n_{l_1}(\epsilon_1) + n_{l_2}(\epsilon_2)\right]\right\};$$

+ kinetic equation for occupation function $n_l(\epsilon)$

<u>*Stability of metallic phase*</u>

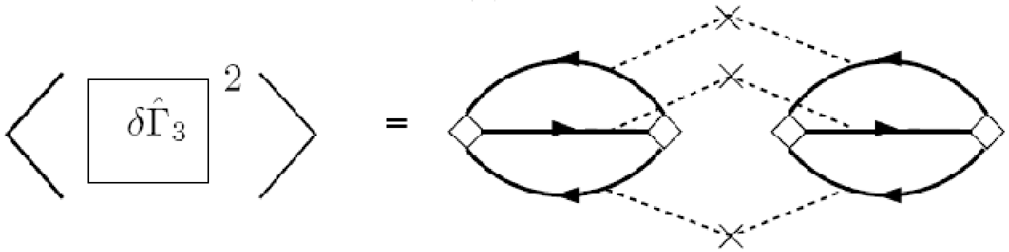
Assume $\Gamma_{in}(\epsilon)$ ***is Gaussian:***



$$\left(\left\langle \Gamma^{(in)} \right\rangle = \pi\lambda^2 MT \right)^2$$

⇓

a)

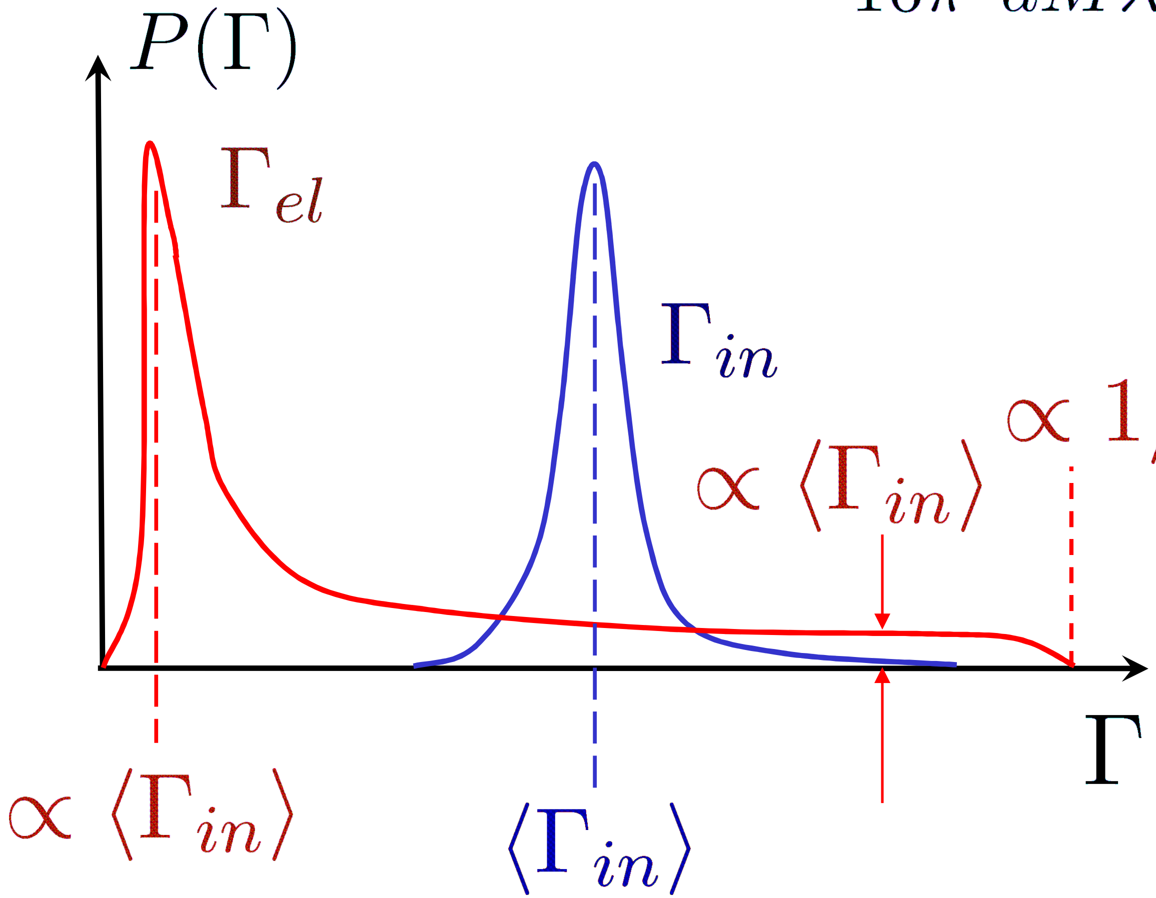


$$\left\langle \left(\delta\Gamma^{(in)} \right)^2 \right\rangle = \frac{\pi\lambda^4 M \delta_\zeta^2 T}{36 \langle \Gamma^{(in)} \rangle}$$

$$T \gtrsim T_{in} \equiv \frac{\delta_\zeta}{6\pi\lambda M}$$

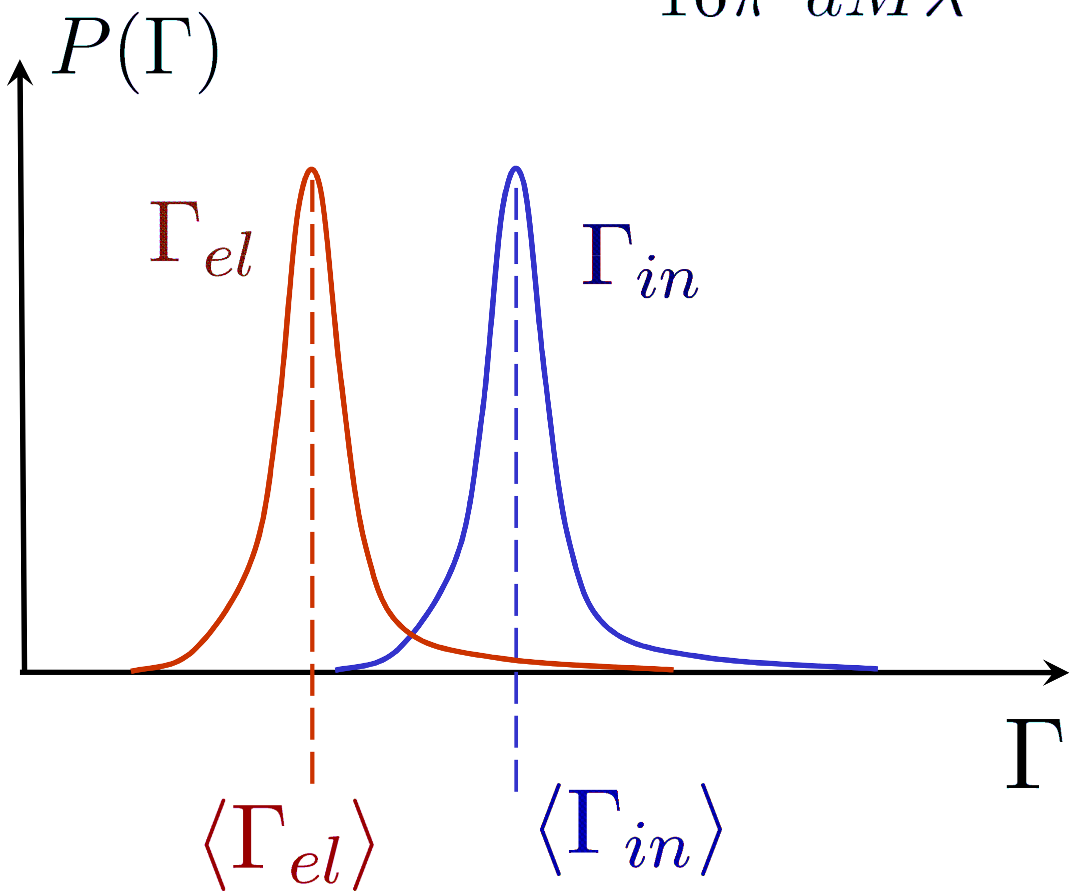
“Non-ergodic” metal [discussed first in AGKL,97]

$$T_{in} \lesssim T \lesssim T_{el} = \frac{\delta_\zeta}{16\pi^2 d M \lambda^2} \simeq \frac{T_{in}}{\lambda}$$



Drude metal

$$T \gtrsim T_{el} = \frac{\delta_\zeta}{16\pi^2 d M \lambda^2} \simeq \frac{T_{in}}{\lambda}$$

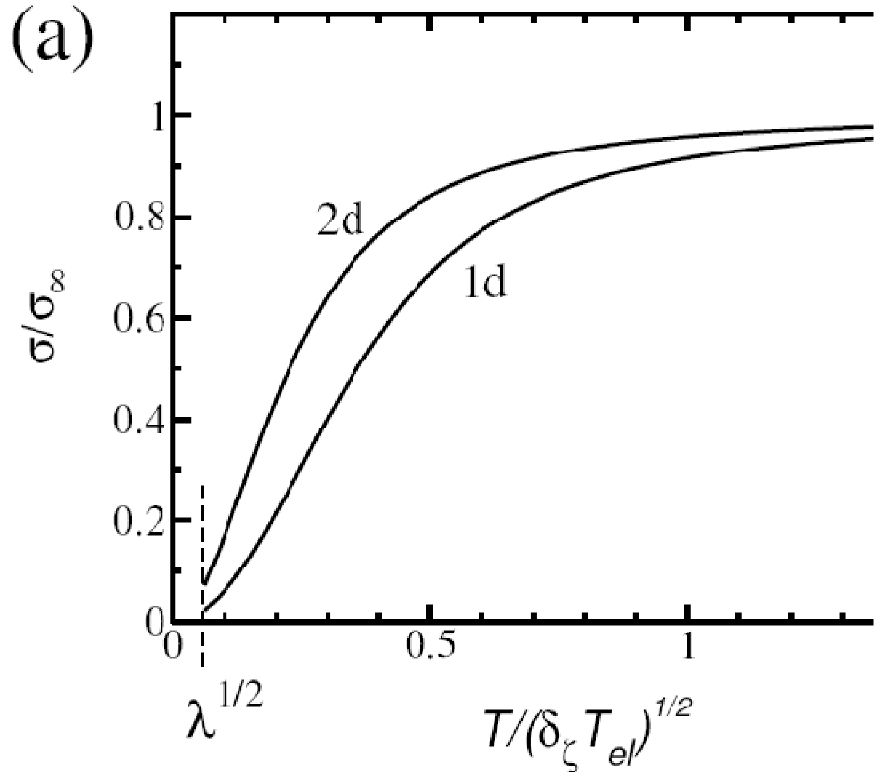


Kinetic Coefficients in Metallic Phase

$$\sigma_\infty \equiv \frac{2\pi e^2 I^2 \zeta_{loc}^{2-d}}{\hbar}$$

$$\sigma(T \gg \sqrt{\delta_\zeta T_{el}}) \approx \sigma_\infty \left(1 - \frac{2}{3}\,\frac{\delta_\zeta T_{el}}{T^2}\right)$$

$$\sigma(T \ll \sqrt{\delta_\zeta T_{el}}) = \sigma_\infty \frac{\pi}{4} \left(\frac{T^2}{\delta_\zeta T_{el}}\right)$$

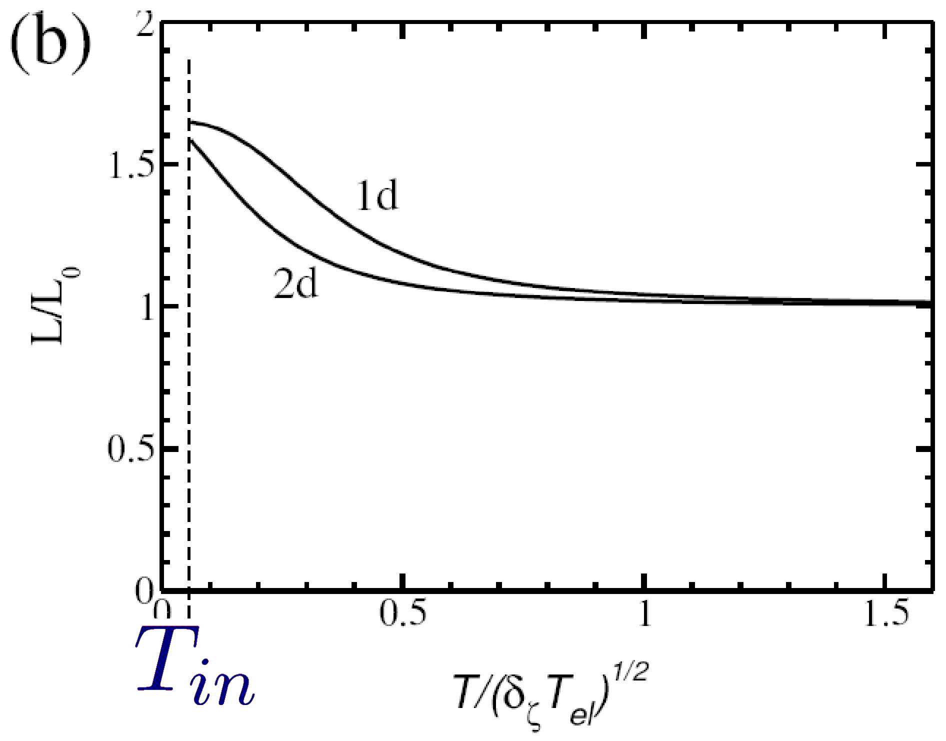


T_{in}

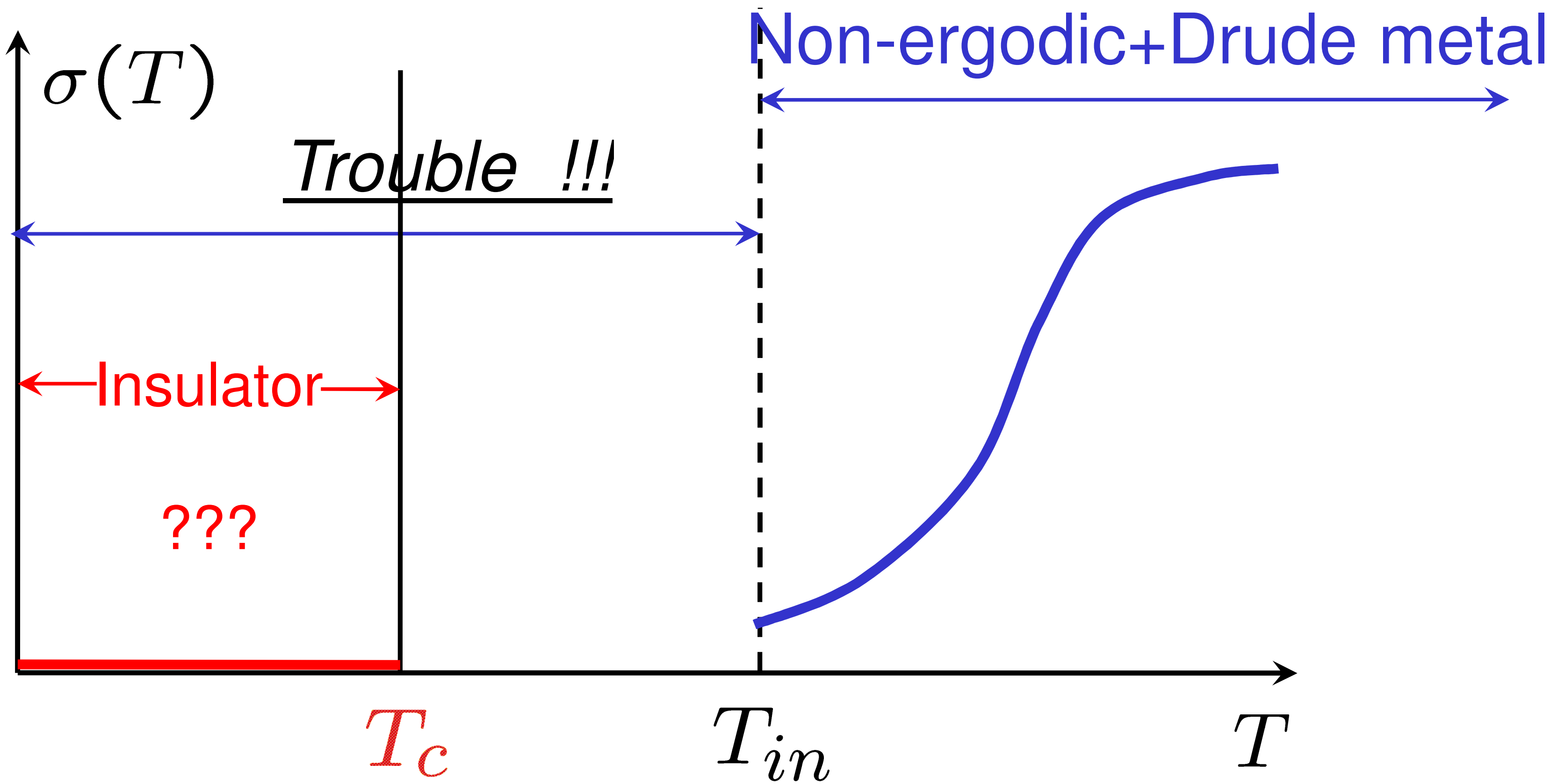
Kinetic Coefficients in Metallic Phase

Wiedemann-Frantz law ?

$$\frac{\mathrm{L}(T)}{\mathrm{L}_0} \equiv \frac{3e^2\kappa(T)}{\pi^2\sigma(T)T} = \begin{cases} 1 + 0.3\left(\frac{\delta_\zeta T_{el}}{T^2}\right), & T \gg \sqrt{\delta_\zeta T_{el}}, \\ \frac{192\mathrm{G}^2}{\pi^4} \approx 1.65\ldots, & T \ll \sqrt{\delta_\zeta T_{el}}. \end{cases}$$



So far, we have learned:



Stability of the insulator

Nonlinear integral equation with random coefficients

$$\Gamma_l(\epsilon) = \Gamma_l^{(el)}(\epsilon) + \Gamma_l^{(in)}(\epsilon) + \eta;$$

$$\Gamma_l^{(el)}(\epsilon, \boldsymbol{\rho}) = \pi I^2 \delta_\zeta^2 \sum_{l_1, \boldsymbol{a}} A_{l_1}(\epsilon, \boldsymbol{\rho} + \boldsymbol{a});$$

$$\Gamma_l^{(in)}(\epsilon) = \pi\lambda^2\delta_\zeta^2 \sum_{l_1,l_2,l_3} Y_{l_1,l_2}^{l_3,l} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 A_{l_1}(\epsilon_1) A_{l_2}(\epsilon_2) A_{l_3}(\epsilon_3) \delta(\epsilon - \epsilon_1 - \epsilon_2 + \epsilon_3) F_{l_1,l_2;l_3}^{\Rightarrow}(\epsilon_1, \epsilon_2; \epsilon_3);$$

$$A_l(\epsilon) = \frac{\pi^{-1}\Gamma_l(\epsilon)}{[\epsilon - \xi_l]^2 + [\Gamma_l(\epsilon)]^2}$$

Notice: $\Gamma(\epsilon) = 0;$ **for** $\eta = 0$ **is a solution**

Linearization:

$$A_l(\epsilon) \approx \delta(\epsilon - \xi_l) + \frac{\Gamma_l(\epsilon)}{\pi(\epsilon - \xi_l)^2}$$

# of interactions

# of hops in space

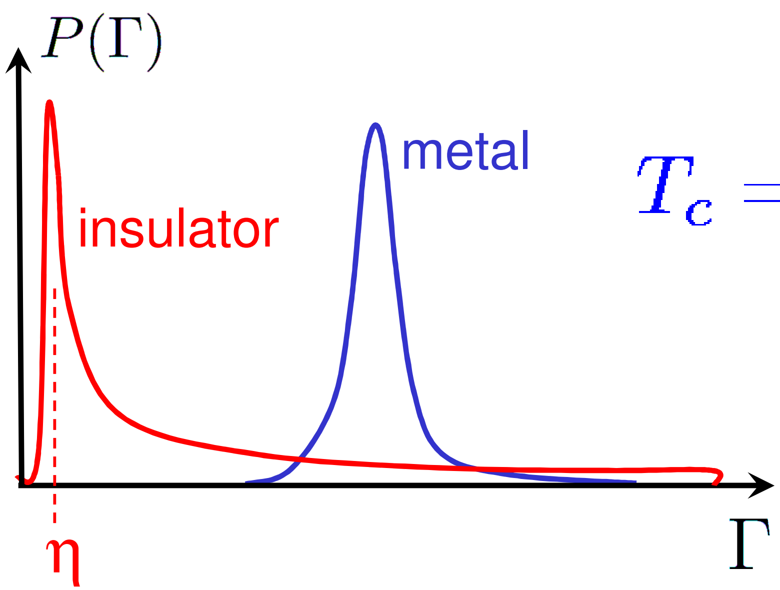
$$\Gamma = \sum_{n,m} \Gamma^{n,m}$$

$$P(\Gamma^{n,m}) = \sqrt{\frac{\gamma^{n,m}}{\pi \left[\Gamma^{n,m}\right]^3}} \exp\left(-\frac{\gamma^{n,m}}{\Gamma^{n,m}}\right)$$

Recall:

$$\gamma^{n,m} \leq \eta \left(\frac{T}{T_c}\right)^n$$

$$T_c = \frac{\delta_\zeta}{12\lambda M |\ln \lambda|} \left[1 + \mathcal{O}\left(\lambda M \ln I\right)\right]$$

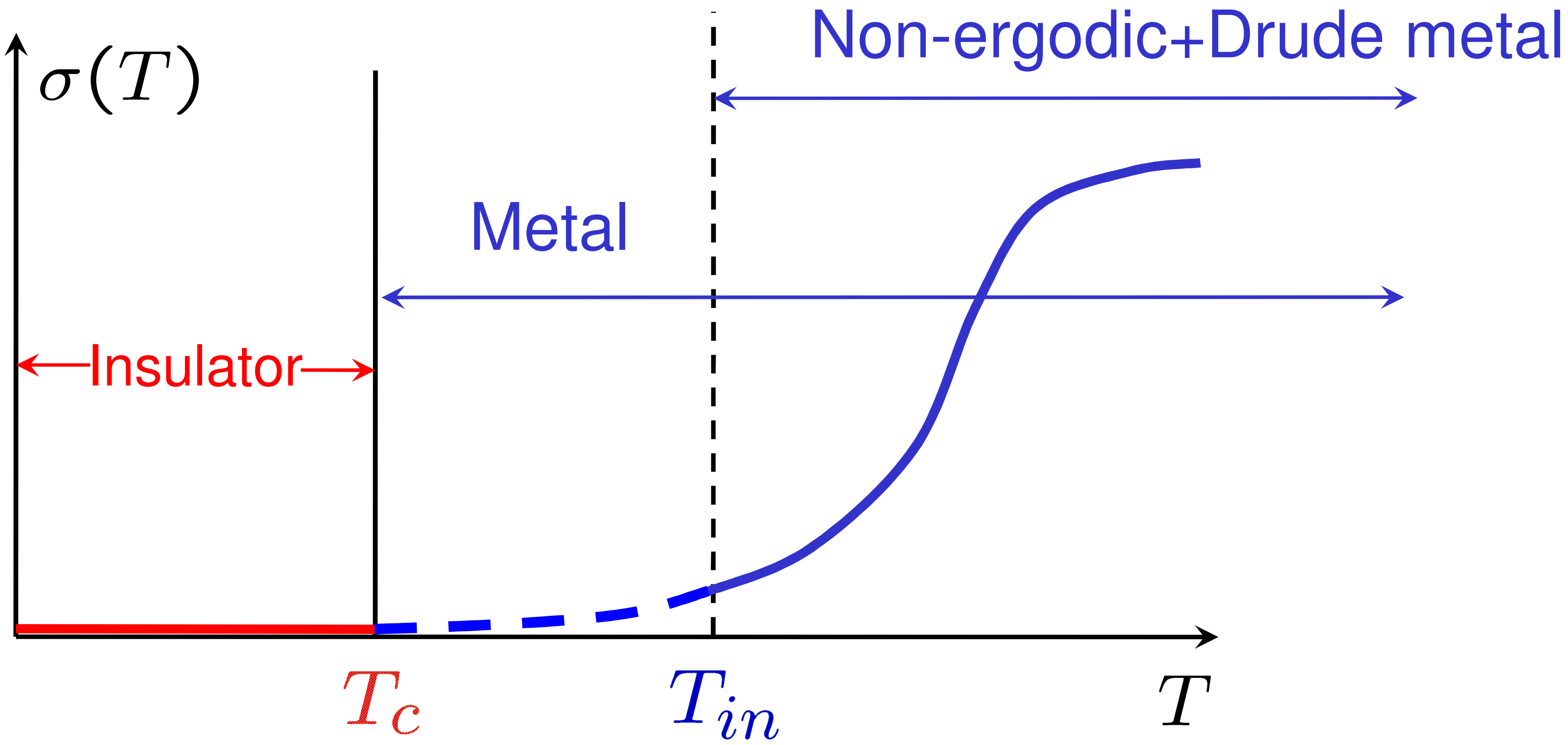


probability distribution
for a fixed energy

$T < T_c$ **STABLE**

$T > T_c$ **unstable**

So, we have just learned:



$$T_c = \frac{\delta_\zeta}{12\lambda M|\ln\lambda|} \qquad T_{in} = \frac{\delta_\zeta}{6\pi\lambda M}$$

Extension to non-degenerate system

$$T_c \gg \epsilon_F$$

$$\hat{H}_{int} = \frac{b}{4} \int d^d \boldsymbol{r} : \left(\hat{\psi}^\dagger \hat{\psi} \right)^2 :, \quad \text{bosons}$$

$$T_c \simeq \frac{\delta_\zeta^2(T_c)}{b n_0}; \quad \text{if} \quad \frac{d\zeta(\epsilon)}{d\epsilon} > 0$$

For 1D it leads to:

$$\frac{\hbar^2}{m\zeta(T_c)^2} \simeq b n_0;$$

I.A. and B.L. Altshuler , unpublished (2008)

Instead of conclusion

Estimate for the transition temperature for general case

1) Start with T=0;

2) Identify elementary (one particle) excitations and prove that they are localized.

3) Consider a one particle excitation at finite T and the possible paths of its decays:

$$¢_t(T_c)\ '\ \ U(T_c)N_1(T_c)$$

Energy mismatch

Interaction matrix element

# of possible decay processes of an excitations allowed by interaction Hamiltonian;