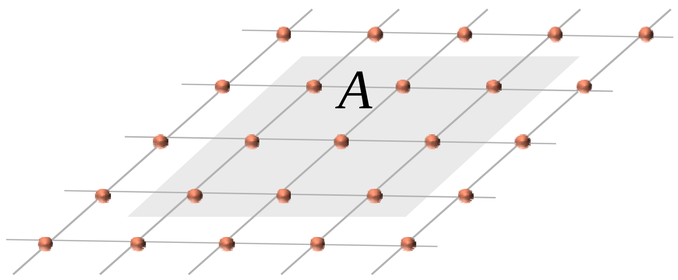
Bulk-boundary correspondence in spin lattices

Work with N. Schuch (Aachen), D. Perez-Garcia (Madrid), D. Poilblanc (Tolouse)
+ F. Verstraete (Vienna), J. Haegeman (Gent)

Theory Seminar,
TECHNION, December 5th, 2012

SPIN LATTICES

- Spins on a lattice in 2D at zero temperature:



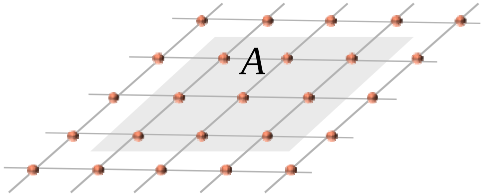
- Many-body state: $|\Psi\rangle$
- Parent Hamiltonian (local)

$$H|\Psi\rangle = E_0|\Psi\rangle$$

- Reduced state in region A:

$$\rho_A = \mathrm{tr}\left[|\Psi\rangle\langle\Psi|\right]$$

SPIN LATTICES



- Area law: (Sredniky 93):

$$S(\rho_A) \sim N_{\partial A}$$

degrees of freedom $\prec$ # particles at boundary

- Entanglement spectrum: (Li and Haldane, 2008; Peschel, Kitaev and Preskill):

$$\sigma(H_A) \quad \text{where} \quad \rho_A = e^{-H_A}$$

The low energy sector has the same structure as that for a lower dimensional theory (edge states)

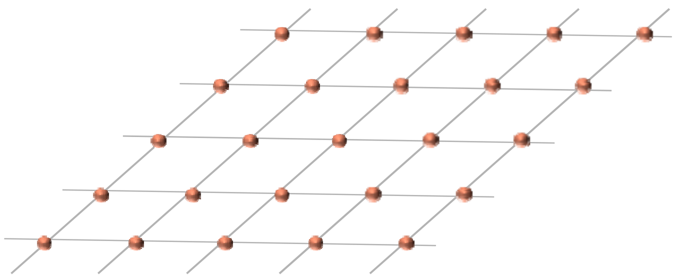
- THE (REDUCED) STATE OF THE BULK CAN BE DESCRIBED BY A LOWER DIMENSIONAL THEORY

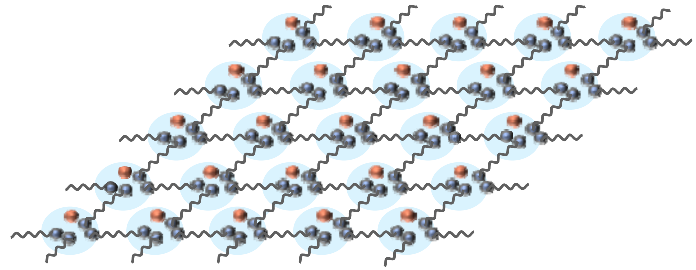
- THAT THEORY IS SOMEHOW RELATED TO THE BOUNDARY OF THE REGION

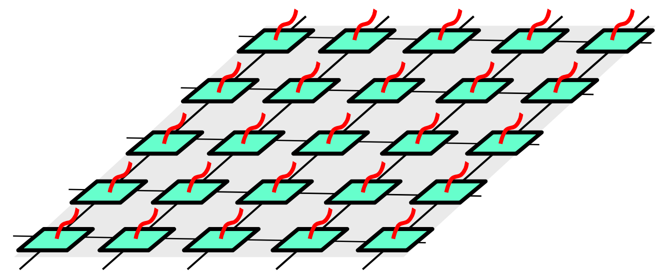
PROJECTED ENTANGLED-PAIR STATES

(Verstraete and IC, 2004)

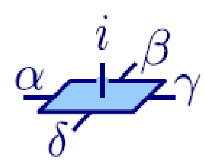
$|\Psi\rangle$







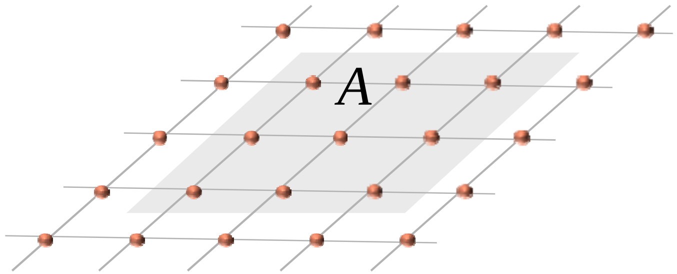
$$|\Psi\rangle \in H^{\otimes N} \quad \leftrightarrow \quad A^{i_n}_{\alpha\beta\gamma\delta}$$

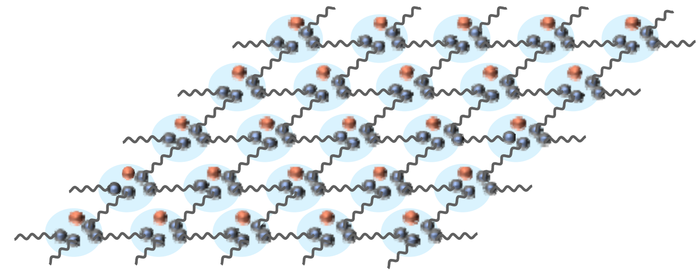


PEPS give a natural playground to investigate this subject

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

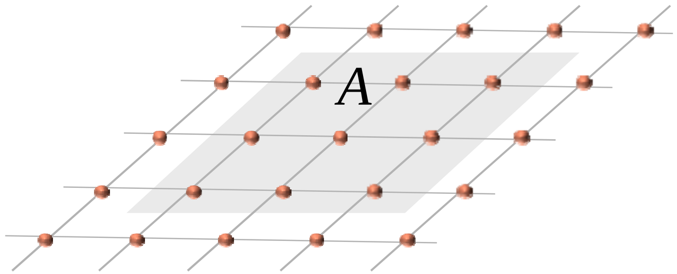
IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)

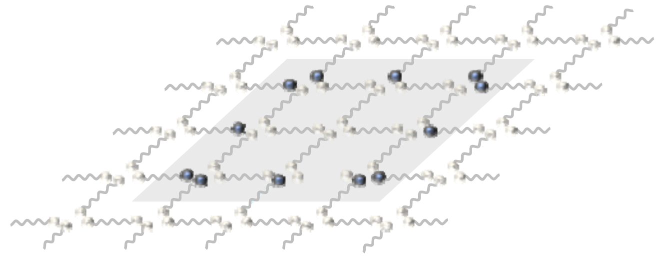




PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)





- The theory corresponds to the auxiliary particles living in the boundary

- Isommetry between the spins in the bulk and the auxiliary ones in the boundary

$$\sigma_{\partial A} = U \rho_A U^{\dagger}$$

↑ isommetry

- It „compresses“ the degrees of freedom
- Conserves the spectrum
- Allows to determine expectation values

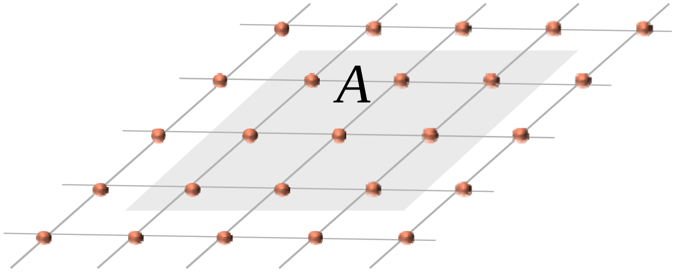
- It defines a BOUNDARY HAMILTONIAN

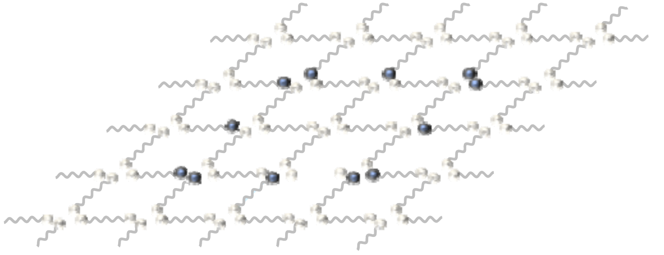
$$\sigma_{\partial A} = e^{-H_{\partial A}}$$

- Has the same entanglement spectrum $\sigma(H_{\partial A}) = \sigma(H_A)$
- It can be easily determined (exactly or approximately)

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)





What can we say starting from the boundary Hamiltonian?
(beyond the entanglement spectrum)

- Is the Hamiltonian local?
- What are its symmetries, and how are they related to those of H?
- How do topological properties manifest themselves?
- What happens in quantum phase transitions?
- How general are those predictions?

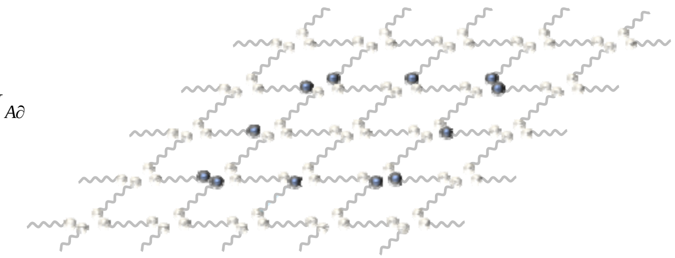
Other approaches: Qi, Katsura, and Ludwig, 2012, Dubail, Read, and Rezayi, 2012

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)

- Results:

$$\sigma_{\partial A} = e^{-H_{A\partial}}$$



- Symmetries: The boundary Hamiltonian inherits the symmetries

$$u_g \,|\Psi\rangle = e^{i\theta_g}\,|\Psi\rangle \quad \Rightarrow \quad U_g H_{\partial A} U_g^{\dagger} = H_{\partial A}$$

- Locality:

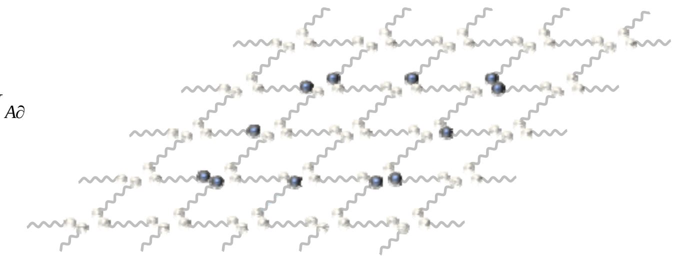
  - For gapped systems, it is local
  - For critical systems, it becomes non-local

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)

- Implications:

$$\sigma_{\partial A} = e^{-H_{A\partial}}$$



- Entanglement spectrum corresponds to CFT theories:

  -Take a problem with su(2) symmetry and such that the boundary particles have semi-integer reps (eg, spin ½)
  - The boundary Hamiltonian is (close to) Heisenberg.

- Quantum phase transitions:

  -They are reflected in the boundary Hamiltonian

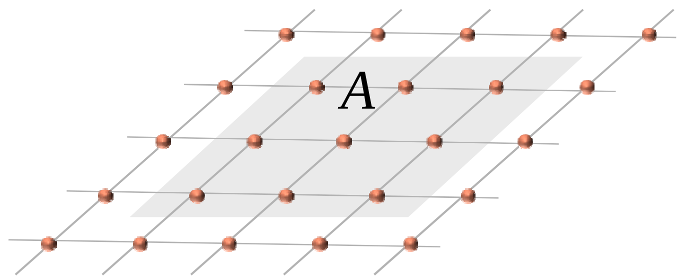
- Contraction of PEPS:

  - One needs to express the boundary density operator as a MPO
  - Thermal states of local Hamiltonians can be (efficiently) written as MPO
  - For gapped system, the boundary density operator is such a thermal sate

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

This talk: Gapped topological phases in 2D

Schuch, Poilblanc, IC, Perez-Garcia, arXiv:1210.5601

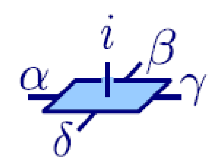


PROPERTIES

- Boundary state
- Boundary Hamiltonian

EXAMPLES

- Toric code (Kitaev)
- RVB states
- Phase transitions



- Results:
  - The boundary theory develops an extra symmetry

$$\sigma_{\partial A} = U_g \sigma_{\partial A} U_g^{\dagger}$$

- In general, the boundary operator is block diagonal $\sigma_{\partial A} = \sigma_{\partial A}^{1} \oplus \sigma_{\partial A}^{2} \oplus \ldots$
- The projector, P_i, in each subspace is higly non-local

  - The boundary Hamiltonian splits

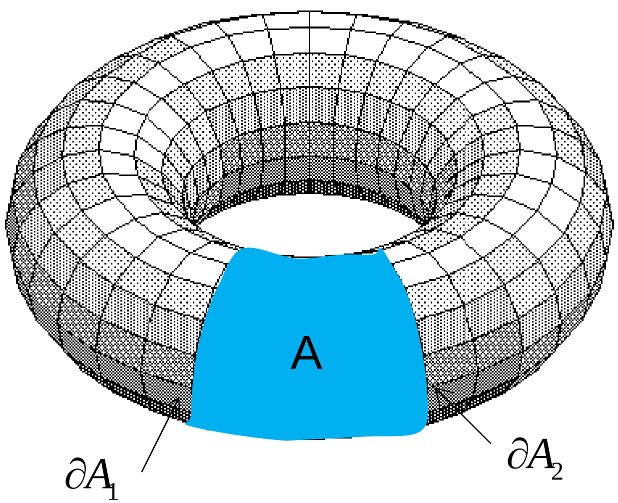
$$H_{\partial A} = H_{\partial A}^{\text{topo}} + H_{\partial A}^{\text{non-universal}}$$

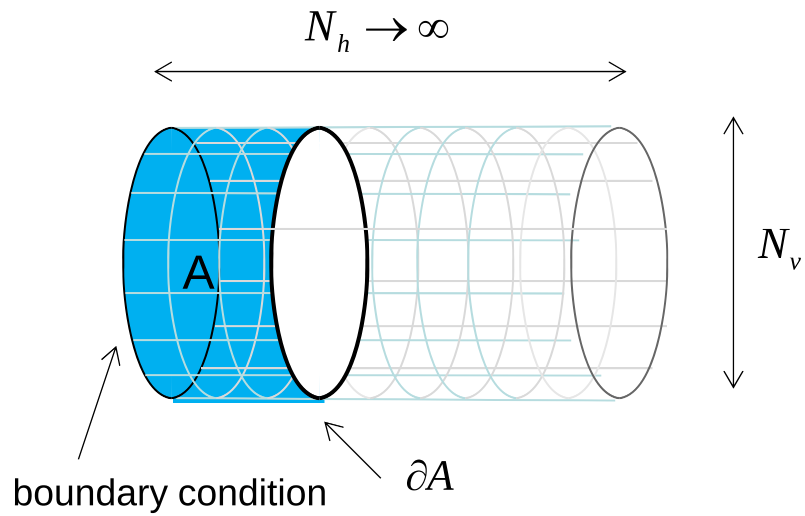
- $H_{\partial A}^{\text{topo}}$ is universal (only depends on the boundary conditions): $H_{\partial A}^{\text{topo}} = \oplus c_i P_i$
- $H_{\partial A}^{\text{non-universal}}$ is local and depends on the details of the state (but not on the boundary conditions)

  - Phase transition

- $H_{\partial A}^{\text{non-universal}}$ becomes non-local
- It can eventually compensate the universal part $H_{\partial A}^{\text{topo}}$

- Setup:
  - Cylinder or Torus



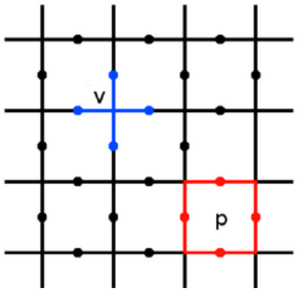


$$\sigma_{\partial A_1 \partial A_2}$$

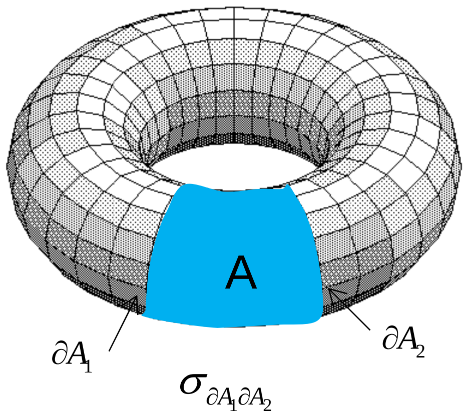
$$\sigma_{\partial A} = \mathrm{tr}\left[X_{\partial A_1} \sigma_{\partial A_1 \partial A_2} \right]$$

BOUNDARY THEORY
TORIC CODE

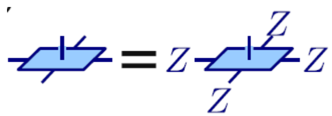
- Ground state of: $H = -\sum_p H_p - \sum_v H_v$
  - Take one of the four ground states
    (the other can be easily obtained from that)
  - Auxiliary particles: qubits (D=2)



BOUNDARY THEORY
TORIC CODE



- Symmetry:



$$\sigma_{\partial A_1 \partial A_2} = \left[Z^{\otimes N_v} \otimes Z^{\otimes N_v} \right] \sigma_{\partial A_1 \partial A_2} \left[Z^{\otimes N_v} \otimes Z^{\otimes N_v} \right]$$

- Projectors: $P_e = \frac{1}{2}\left(1 + Z^{\otimes N_v}\right)$ even parity

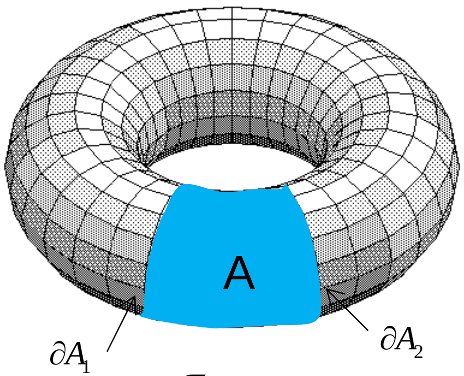
  $P_o = \frac{1}{2}\left(1 - Z^{\otimes N_v}\right)$ odd parity

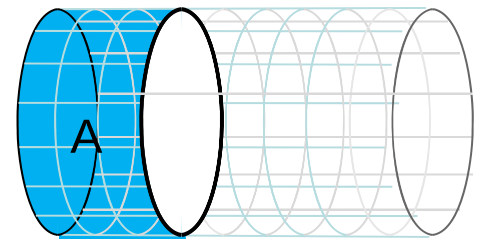
- Boundary state:

$$\sigma_{\partial A_1 \partial A_2} = P_e \otimes P_e + P_o \otimes P_o$$

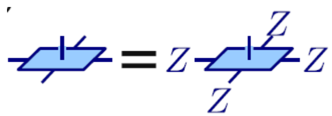
-Total even parity
- Non-local

BOUNDARY THEORY
TORIC CODE





- Symmetry:



$$\sigma_{\partial A_1 \partial A_2} = \left[Z^{\otimes N_v} \otimes Z^{\otimes N_v} \right] \sigma_{\partial A_1 \partial A_2} \left[Z^{\otimes N_v} \otimes Z^{\otimes N_v} \right]$$

- Projectors: $P_e = \frac{1}{2}\left(1 + Z^{\otimes N_v}\right)$ even parity

  $P_o = \frac{1}{2}\left(1 - Z^{\otimes N_v}\right)$ odd parity

- Boundary state:

$$\sigma_{\partial A_1 \partial A_2} = P_e \otimes P_e + P_o \otimes P_o$$

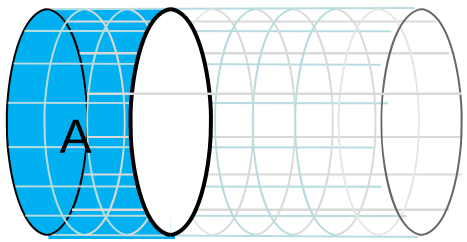
$$\Rightarrow \quad \sigma_{\partial A} = \mathrm{tr}\left[X_{\partial A_1} \sigma_{\partial A_1 \partial A_2} \right] = p_e P_e + p_o P_o$$

-Total even parity
- Non-local

(note top EE requires $p_e = 0$)

BOUNDARY THEORY

TORIC CODE



$$\sigma_{\partial A} = \mathrm{tr}\left[X_{\partial A_1} \sigma_{\partial A_1 \partial A_2} \right] = p_e P_e + p_o P_o$$

- Symmetry: $\sigma_{\partial A} = Z^{\otimes N_v} \sigma_{\partial A} Z^{\otimes N_v}$

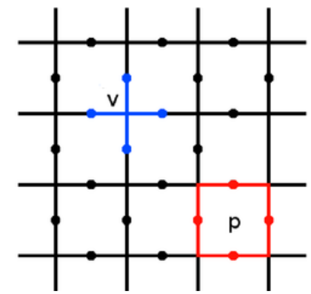
- Projectors: $P_e = \frac{1}{2}\left(1 + Z^{\otimes N_v}\right)$ even parity

  $P_o = \frac{1}{2}\left(1 - Z^{\otimes N_v}\right)$ odd parity

- Boundary Hamiltonian: $H_{\partial A} = H_{\partial A}^{\text{topo}} = -\log(p_e) P_e - \log(p_o) P_o$

  - The values of $p_{e,o}$ depend on the chosen boundaries
  - It is highly non-local (like the projectors)
  - There is no non-universal part (it is a fixed point of the RG flow)

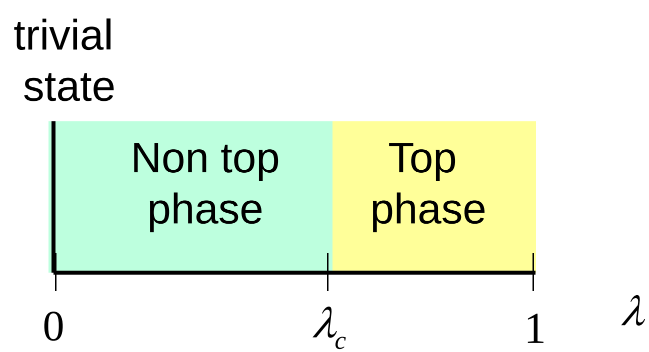
- Deformed Kitaev model:



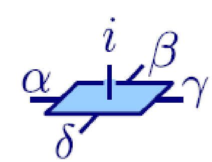
  - Kitaev: $H = -\sum_p H_p - \sum_v H_v$
  - Deformed Kitav model (Castelnovo and Chamon, PRL 2008)

$$H(\lambda) = -\sum_p H_p(\lambda) - \sum_v H_v(\lambda)$$

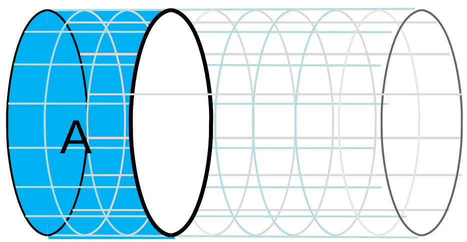
$$H_x(\lambda) = (|0\rangle\langle 0| + \lambda^{-1}|1\rangle\langle 1|)^{\otimes 4} H_x (|0\rangle\langle 0| + \lambda^{-1}|1\rangle\langle 1|)^{\otimes 4}$$

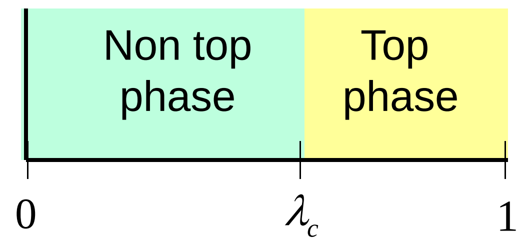


The ground state is a PEPS (with D=2) for all values of λ



- Deformed Kitaev model:





- Conjecture: $H_{\partial A} = H_{\partial A}^{\text{topo}} + H_{\partial A}^{\text{non-universal}}$

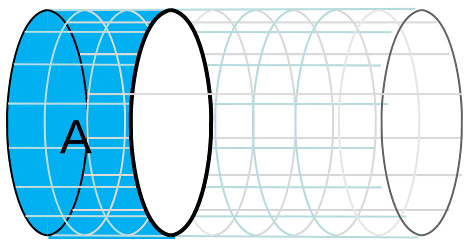
  where $H_{\partial A}^{\text{topo}} = -\log(p_e)P_e - \log(p_o)P_o$ is the one calculated for λ=1 (toric code)

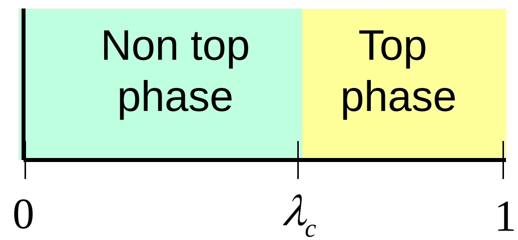
- Calculation:

  1. Fix the boundary condition (determines $p_{e,o}$).
  2. Determine the boundary state: $\sigma_{\partial A}$
  3. Compute the boundary Hamiltonian: $H_{\partial A} = -\log\left(\sigma_{\partial A}\right)$
  4. Substract the universal topo Hamiltonian: $H_{\partial A}^{\text{non-universal}} = H_{\partial A} - H_{\partial A}^{\text{topo}}$
  5. Determine the „interaction lenght of the non-universal part“

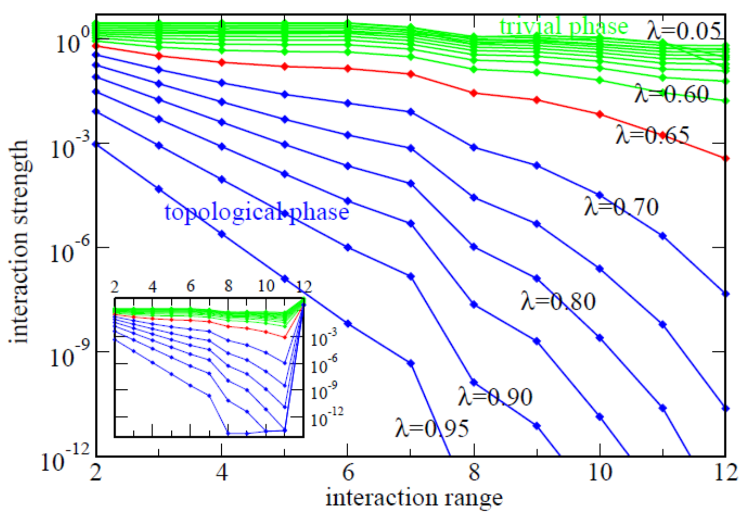
BOUNDARY THEORY
PHASE TRANSITIONS

- Deformed Kitaev model:





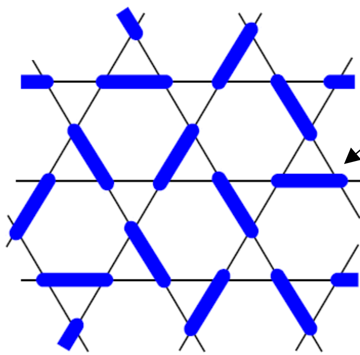
$$H_{\partial A}^{\text{non-universal}} = H_{\partial A} - H_{\partial A}^{\text{topo}}$$



- Lattice: $\infty \times 12$
- Topological pase:
  Interaction lenght decreases exponentially
- Non-Topological pase:
  Long-range interactions

- RVB states in a Kagome lattice:

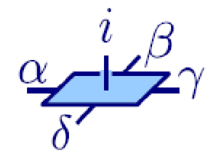
VBS



$|0,1\rangle - |1,0\rangle$

$$|RVBS\rangle = \sum |VBS\rangle$$

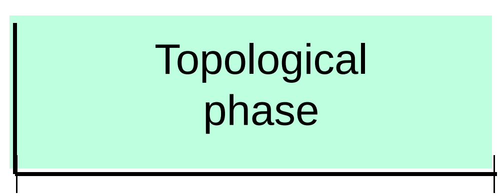
$A^{i_n}_{\alpha\beta\gamma\delta}$



$$A^0_{0222} = A^0_{2022} = -A^0_{2212} = -A^0_{2221} = 1$$

$$A^1_{1222} = A^1_{2122} = -A^1_{2202} = -A^1_{2220} = 1$$

- We can find a local parent Hamiltonian, H.
- We can find a deformation: $A^{i_n}_{\alpha\beta\gamma\delta}(\theta) \;\; \Rightarrow |\Psi(\theta)\rangle$

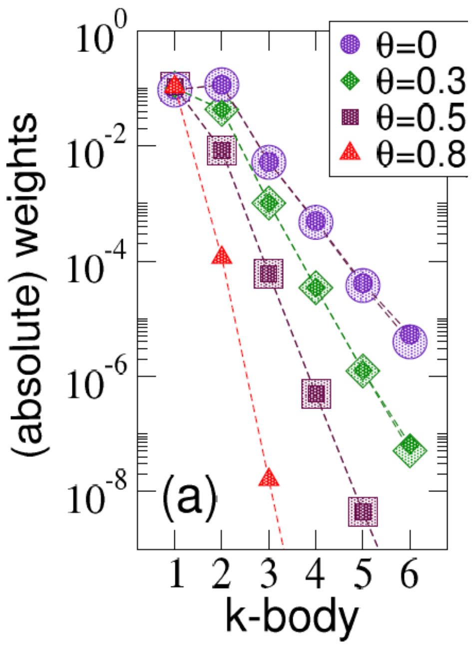


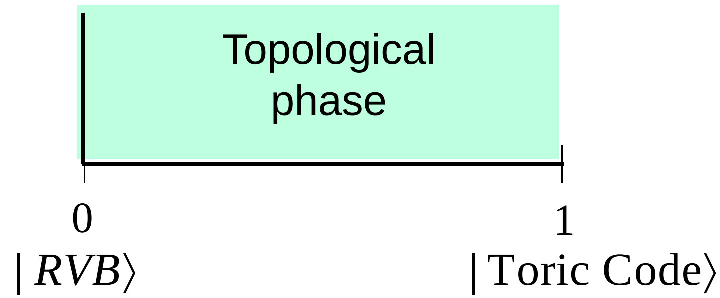
0 $|RVB\rangle$

1 $|\text{Toric Code}\rangle$

BOUNDARY THEORY
PHASE TRANSITIONS

- RVB states in a Kagome lattice:





- We calculate the non-universal part.

$$H_{\partial A}^{\text{non-universal}} = H_{\partial A} - H_{\partial A}^{\text{topo}}$$

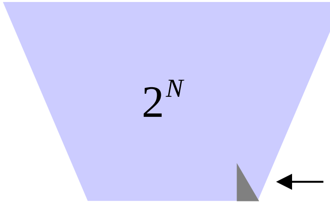
- It is local: no phase transition

BOUNDARY THEORY

How general are those results?

- Conclusions are based on examples
- Believe that is generic for PEPS
- PEPS describe gapped phases

    - What about chiral topological theories?



physically relevant

- Parent Hamiltonian for Laughlin spin state in a lattice

Anne Nielsen, IC, German Sierra, arXiv 1201.3096

- Build iMPS with correlators of a CFT

  Sierra and IC, PRB **81**, 104431 (2010)

- Use null vectors of the CFT to build a parent Hamiltonian for a spin system

  Nielsen, IC, Sierra, J. Stat. Mech. (2011), P11014 for $SU(2)_k$ WZW theory

$$Q_a^i \,|\Psi\rangle = 0 \;\Longrightarrow\; \underbrace{\sum_{i,a} Q_a^{i\dagger} Q_a^i}_{H} \,|\Psi\rangle = 0$$

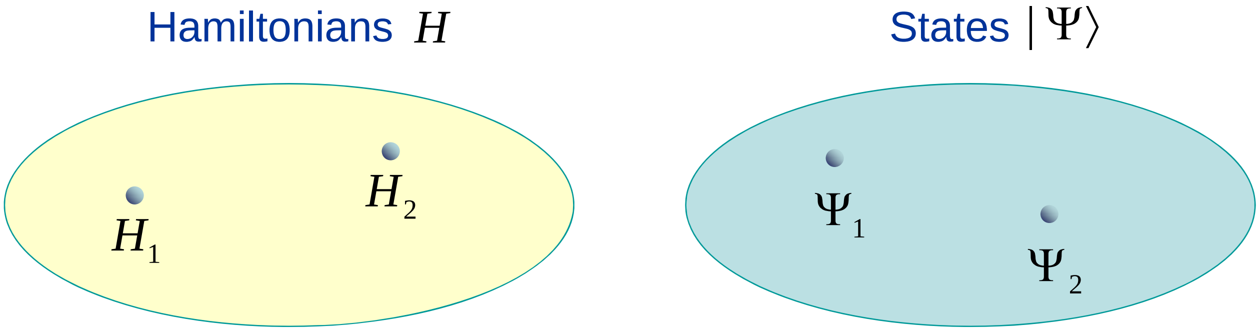
- Investigate topological properties by EE and ES

  (proof that for two-leg ladder, EH is invariant under Yangian trafo)

GAPPED PHASES AT T=0 in 1-dimension

Schuch, Perez-Garcia, Cirac, Phys. Rev. B 84, 165139 (2011), arxiv:1010.3732
Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011), arxiv:1008.3745

- Phases:

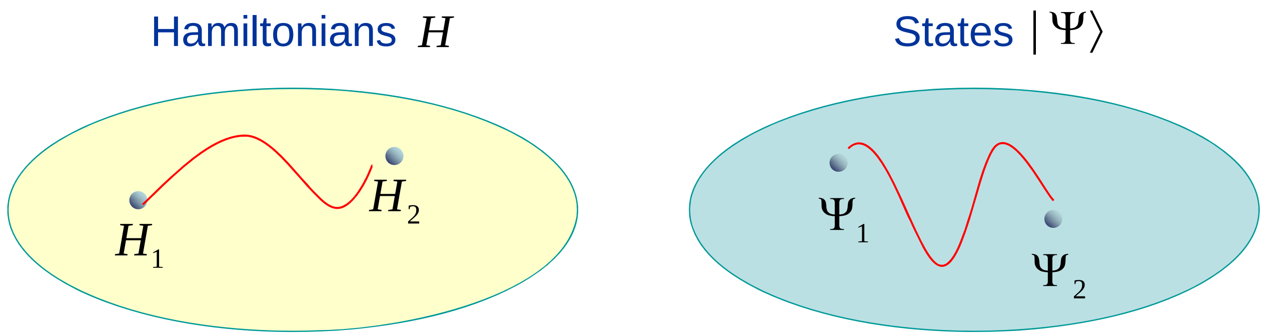


Ψ_1 and Ψ_2 are in the same phase if they can be smoothly connected

2. GAPPED PHASES

2.1. CLASSIFICATION

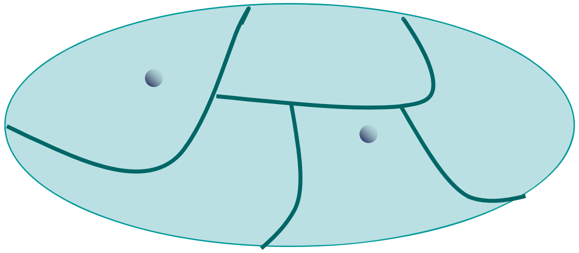
- Phases:



H_1 and H_2 are gapped (above the ground subspace), then $H(\gamma)$ is gapped

- Phases:

States $|\Psi\rangle$

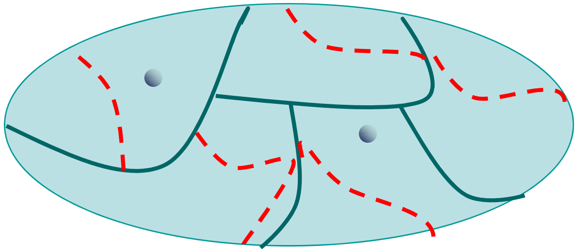


along the boundaries, there are quantum phase transitions

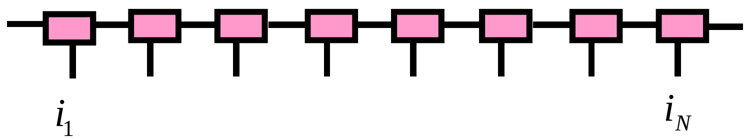
- Symmetries: $H_{1,2} = u_g^{\otimes N} H_{1,2} u_g^{\otimes N \dagger}$ $\quad |\Psi_{1,2}\rangle = u_g^{\otimes N} |\Psi_{1,2}\rangle$

We may impose that the path $H(\gamma)$ respects the symmetries

States $|\Psi\rangle$

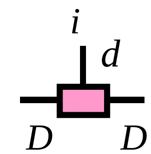


- Definition:



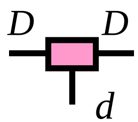
$$c^{i_1, i_2, \ldots} = \mathrm{tr}\left[A_1^{i_1} A_2^{i_2} \ldots A_N^{i_N} \right]$$

A^i



- Parent Hamiltonians:

$A^i_{\alpha\beta}$



+ conditions

- $H = \sum_n P_{(n)}$

  $P_{(n)} | \Psi \rangle = 0$

- Unique, gapped, degenerate

- Symmetries:

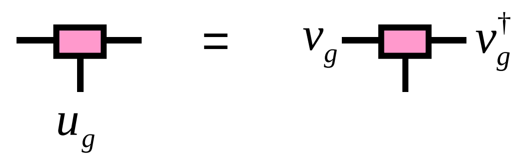
STANDARD

$$u_g^{\otimes N} \, | \Psi \rangle = e^{i\phi_g} \, | \Psi \rangle$$

$$g \in G$$

u_g (linear) representation of G

MPS



v_g (projective) unitary representations of G

Symmetries in the state reflect themselves in symmetries of the tensor

D. Perez-Garcia, M. Sanz, C. E. Gonzalez-Guillen, M. M. Wolf, and J. I. Cirac, New J. Phys. 12, 025010 (2010),

2. GAPPED PHASES IN 1D
2.2. MPS

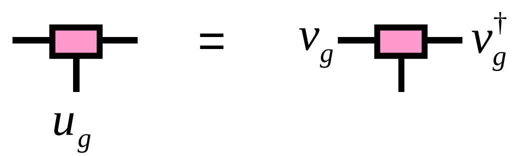
- Symmetries:

STANDARD

$$u_g^{\otimes N}|\Psi\rangle = e^{i\phi_g}|\Psi\rangle$$

$$g \in G$$

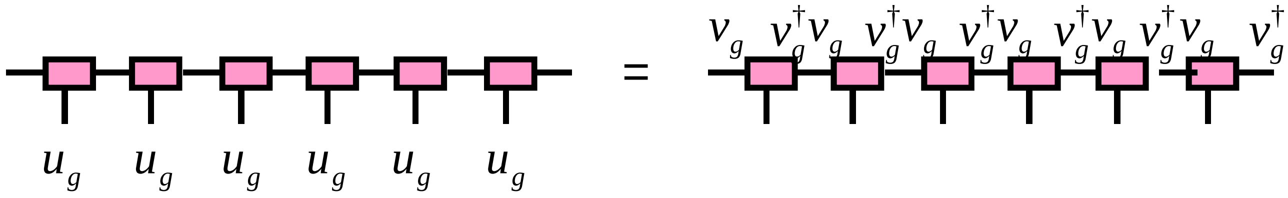
MPS



u_g (linear) representation of G

v_g (projective) unitary representations of G

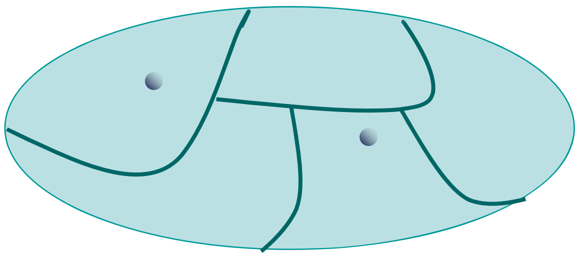
$$u_g^{\otimes N}|\Psi\rangle$$



D. Perez-Garcia, M. Sanz, C. E. Gonzalez-Guillen, M. M. Wolf, and J. I. Cirac, New J. Phys. 12, 025010 (2010),

- No symmetries:

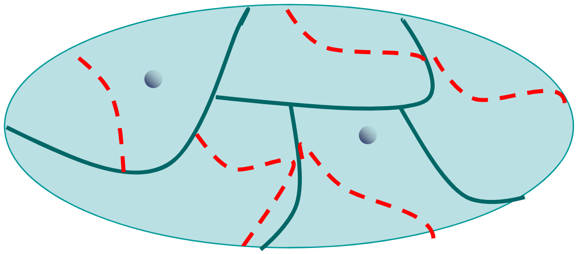
States $|\Psi\rangle$



- The phases are defined in terms of the ground state degeneracy
- In particular, every ground state of a non-degenerate Hamiltonian is in the same phase as the trivial product state.

- Symmetries:

States $|\Psi\rangle$

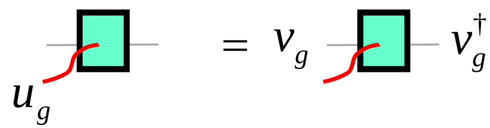


- Non-degenerate ground state:
  The phases are defined in terms of the 2nd cohmology classes of the projective representations of the symmetry group G.

- Degenerate ground state:
  The phases are defined in terms of the 2nd cohmology classes of the induced projective representations of the symmetry group G.

2. GAPPED PHASES IN 1D
2.3. CLASSIFICATION

- Symmetries:



- Linear representation: $u_g u_h = u_{gh}$
- Projective representation: $v_g v_h = e^{i\omega(g,h)} v_{gh}$
  - $v_g \rightarrow v_g e^{i\chi_g}$ (is defined up to a phase)
  - $\omega(g,h)$ is defined up to an eq. relation

The equivalence classes are the cohomology classes

- The cohomology class cannot be changed while keeping analiticity:
- The cohomology class can be (in principle) measured

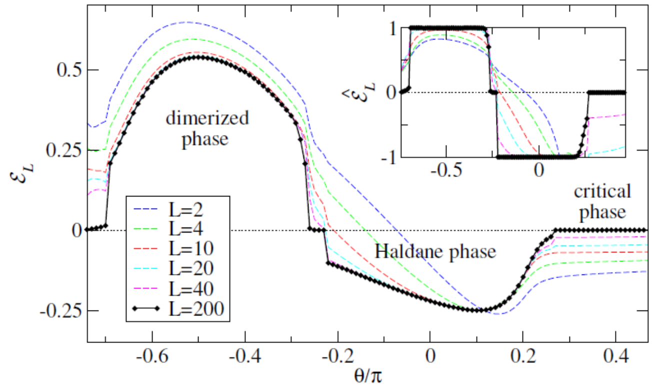
2.4. ORDER PARAMETER

- An order parameter for gapped phases in 1D

Haegeman, Perez-Garcia, IC, Schuch; Phys. Rev. Lett. 109, 050402 (2012)

$$o = \langle \Psi | (u_g^{\otimes N_1} \otimes u_g^{\otimes N_2} \otimes 1^{\otimes N_3}) \mathbf{F}_{13} (1^{\otimes N_1} \otimes u_h^{\otimes N_2} \otimes u_h^{\otimes N_1}) | \Psi \rangle$$

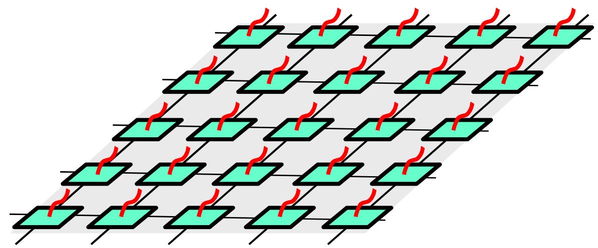
$$H = -\cos(\theta) \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + \sin(\theta) \sum_n \left(\vec{S}_n \cdot \vec{S}_{n+1} \right)^2$$



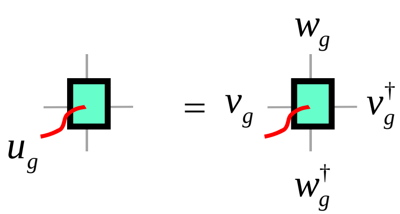
2. GAPPED PHASES

2.5. HIGHER DIMENSIONS

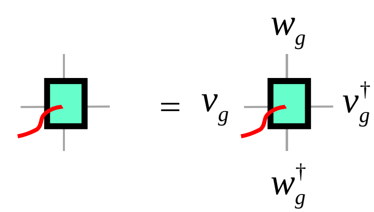
- PEPS:



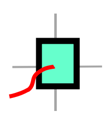
- Symmetries:



- Topology:



- Parent Hamiltonians:

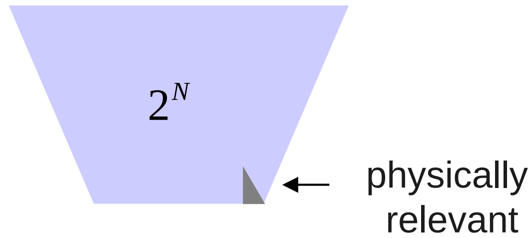


+ conditions

- $H = \sum_n P_{(n)}$
- Unique, ~~gapped,~~ degenerate

- Boundary-bulk correspondence: Schuch, Perez-Garcia, IC (in progress)

SUMMARY and CONCLUSIONS



CONCLUSION:

- Thermal equilibrium and local interaction spins can be efficiently described by PEPS
  - Numerical algorithms
  - New perspective

HERE:

- Area law, entanglement spectrum: bulk-boundary correspondence
- Topological theories:

$$H_{\partial A} = H_{\partial A}^{\text{topo}} + H_{\partial A}^{\text{non-universal}}$$

- Classification of gapped phases in 1D