# CURRENT FLUCTUATIONS IN NON-EQUILIBRIUM SYSTEMS

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## COLLABORATORS

B. Douçot, P.-E. Roche 2004

T. Bodineau 2004-2008

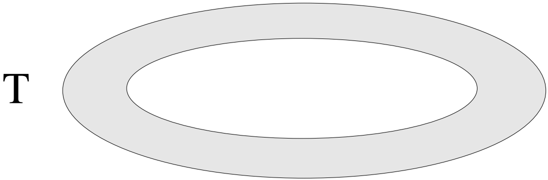
C. Appert, V. Lecomte, F. Van Wijland 2008

A. Gerschenfeld, E. Brunet 2009-2011

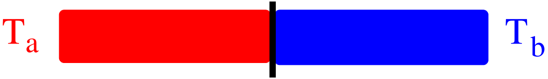
Haifa 2 2011

Current fluctuations in non-equilibrium steady states


Universal fluctuations on a ring



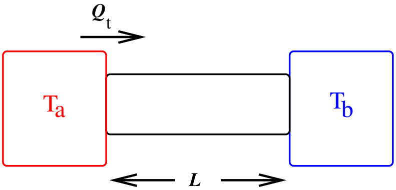

Non-equilibrium initial condition

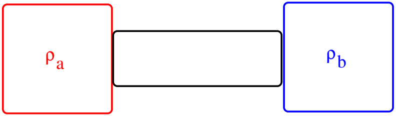


# NON EQUILIBRIUM STEADY STATE

HEAT

PARTICLES




Fick's law

Fourier's law

$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(T_a, T_b)}{L}$$

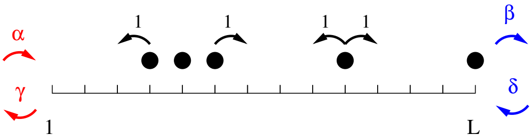
Fluctuation Theorem

$$\log \frac{P(Q_t)}{P(-Q_t)} = (A(T_b) - A(T_a))Q_t$$

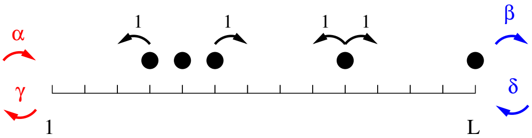
This talk

$$P(Q)?$$

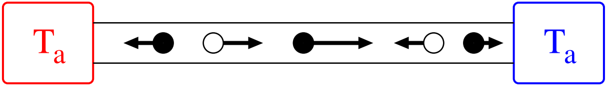
# Symmetric Simple Exclusion Process



# Symmetric Simple Exclusion Process




# One dimensional hard particle gas

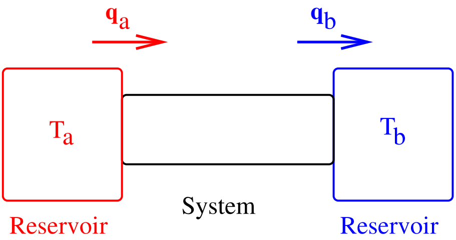


## Detailed balance $T_a = T_b = T$

$$W(\mathcal{C}',\mathcal{C})e^{-\frac{E(\mathcal{C})}{kT}} = W(\mathcal{C},\mathcal{C}')e^{-\frac{E(\mathcal{C}')}{kT}}$$

$$q = E(\mathcal{C}') - E(\mathcal{C}) \qquad W_q(\mathcal{C}',\mathcal{C}) = W_{-q}(\mathcal{C},\mathcal{C}')e^{-\frac{q}{kT}}$$

## Generalized detailed balance $T_a \neq T_b$

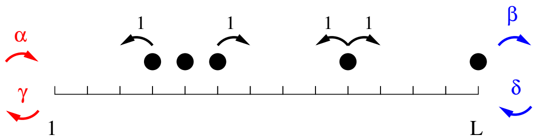



$$q_a - q_b = E(\mathcal{C}') - E(\mathcal{C})$$

$$W_{q_a,q_b}(\mathcal{C}',\mathcal{C}) = W_{-q_a,-q_b}(\mathcal{C},\mathcal{C}')e^{-\frac{q_a}{kT_a}+\frac{q_b}{kT_b}}$$

# SSEP (Symmetric simple exclusion process)

D. Douçot Roche 2004

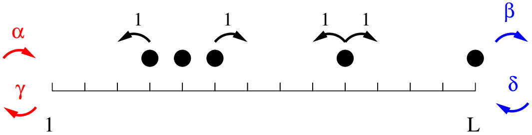



$$\rho_a = \frac{\alpha}{\alpha+\gamma}\ , \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{L}[\rho_a - \rho_b] \qquad \text{Fick's law}$$

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a\rho_b + \rho_b^2)}{3}\right]$$

$$\lim_{t\to\infty} \frac{\langle Q^3(t)\rangle_c}{t} \simeq \frac{1}{L}(\rho_a - \rho_b)\left[1 - 2(\rho_a + \rho_b) + \frac{16\rho_a^2 + 28\rho_a\rho_b + 16\rho_b^2}{15}\right]$$




$$\rho_a = \frac{\alpha}{\alpha+\gamma}\,, \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

D. Douçot Roche 2004

$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{L}[\rho_a - \rho_b] \qquad \text{Fick's law}$$

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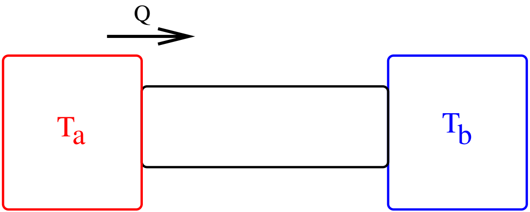
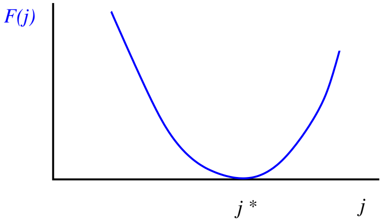
$$\lim_{t\to\infty} \frac{\langle Q^3(t)\rangle_c}{t} \simeq \frac{1}{L}(\rho_a - \rho_b)\left[1 - 2(\rho_a+\rho_b) + \frac{16\rho_a^2 + 28\rho_a\rho_b + 16\rho_b^2}{15}\right]$$

$$\lim_{t\to\infty} \frac{\langle Q^4(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(7\rho_a^2 + \rho_a\rho_b + 7\rho_b^2)}{3} + \frac{32\rho_a^3 + 8\rho_a^2\rho_b + 8\rho_a\rho_b^2 + 32\rho_b^3}{5} - \frac{96\rho_a^4 + 64\rho_a^3\rho_b - 40\rho_a^2\rho_b^2 + 64\rho_a\rho_b^3 + 96\rho_b^4}{35}\right]$$

# CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



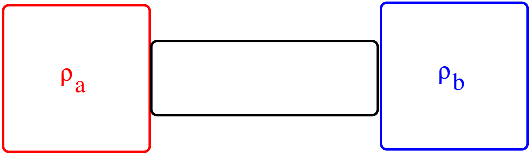

$Q_t$ Energy transferred during time t




$$\text{Pro}\left(\frac{Q_t}{t} = j\right) \sim \exp[-t\ F(j)]$$

Expansion of $F(j)$ near $j^*$ gives all cumulants of $Q_t$

## DIFFUSIVE SYSTEM




Bodineau D. 2004

**One assumes**

- For $\rho_a - \rho_b$ small: $\qquad \frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$
- $\rho_a = \rho_b = \rho$ : $\qquad \frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

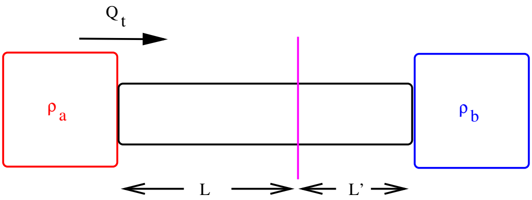
**One can then calculate**

All cumulants of $Q_t$ for arbitrary $\rho_a$ and $\rho_b$

# ADDITIVITY PRINCIPLE

$$\boxed{\mathrm{Pro}\left(\frac{Q_t}{t} = j, \rho_a, \rho_b\right) \sim \exp[-t\ F_{L+L'}(j, \rho_a, \rho_b)]}$$

Bodineau D. 2004




**Additivity**

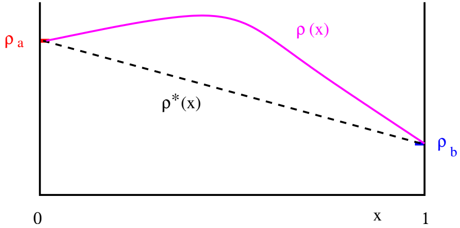
$$P_{L+L'}(Q, \rho_a, \rho_b) \sim \max_{\rho} [P_L(Q, \rho_a, \rho)\ P_{L'}(Q, \rho, \rho_b)]$$

$$\boxed{F_{L+L'}(j, \rho_a, \rho_b) = \min_{\rho} [F_L(j, \rho_a, \rho)\ +\ F_{L'}(j, \rho, \rho_b)]}$$

# VARIATIONAL PRINCIPLE

Bodineau D. 2004

$$F_L(j, \rho_a, \rho_b) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \, \frac{[j\ L\ + \rho'(x)\ D(\rho(x))]^2}{2\sigma(\rho(x))}$$




## Satisfies the fluctuation theorem

$$F(j) - F(-j) = j \int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

Gallavotti Cohen 1995
Evans Searls 1994
.... Kurchan 1998
Lebowitz Spohn 1999

## CONSEQUENCES

$$\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\, \mu(\lambda)}$$

$$\mu(\lambda, \rho_a, \rho_b) = -\frac{K}{L}\left[\int_{\rho_b}^{\rho_a} \frac{D(\rho)\ d\rho}{\sqrt{1+2K\sigma(\rho)}}\right]^2$$

$$\lambda = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sigma(\rho)}\left[\frac{1}{\sqrt{1+2K\sigma(\rho)}} - 1\right]$$

# Non universal cumulants

$$\frac{\langle Q_t \rangle_c}{t} = \frac{1}{L}\mathcal{I}_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{L}\ \frac{\mathcal{I}_2}{\mathcal{I}_1}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{L}\ \frac{3\ (\ \mathcal{I}_3\ \mathcal{I}_1 - \mathcal{I}_2^2\ )}{\mathcal{I}_1^3}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L}\ \frac{3\ (\ 5\ \mathcal{I}_4\ \mathcal{I}_1^2 - 14\ \mathcal{I}_1 \mathcal{I}_2\ \mathcal{I}_3 +\ 9\ \mathcal{I}_2^3\ )}{\mathcal{I}_1^5}$$

where

$$\mathcal{I}_n = \int_{\rho_b}^{\rho_a} D(\rho)\sigma(\rho)^{n-1} d\rho$$

For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$

# MACROSCOPIC FLUCTUATION THEORY

## Diffusive systems

Kipnis Olla Varadhan 89
Spohn 91
⋮
Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001-2002

Evolution of a profile $\rho(x,t)$ for $0 \leq t \leq T$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\}) \sim \exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \, \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

- $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation law)
- $\rho(0,t) = \rho_a \;\; ; \;\; \rho(1,t) = \rho_b$

(For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$)

# TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$F(j) = \frac{1}{L} \lim_{T\to\infty} \min_{\rho(x,t),j(x,t)} \frac{1}{T} \int_0^T dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$ and

$$j\ T = \int_0^T j_t(x) dt$$

- Sufficient condition for the optimal profile to be time independent
- Dynamical phase transition

Bodineau D. 2005-2007

the optimal $\rho_t(x)$ starts to become time dependent

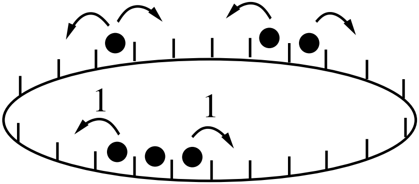
# SSEP ON A RING

Appert D Lecomte Van Wijland 2008

$N$ particles

$L$ sites

$$\rho = \frac{N}{L}$$




$Q_t$ flux through a bond during time $t$

# CUMULANTS OF THE CURRENT FOR THE SSEP ON A RING

$N$ particles and $L$ sites $\qquad \sigma(\rho) = 2\rho(1-\rho) = \frac{2N(L-N)}{L^2}$

$$\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L-1}$$

$$\frac{\langle Q^4\rangle_c}{t} = \frac{\sigma^2}{2(L-1)^2}$$

$$\frac{\langle Q^6\rangle_c}{t} = -\frac{(L^2-L+2)\sigma^3-2(L-1)\sigma^2}{4(L-1)^3(L-2)}$$

$$\frac{\langle Q^8\rangle_c}{t} = \frac{(10L^4-2L^3+27L^2-15L+18)\sigma^4-4(L-1)(11L^2-L+12)\sigma^3+48(L-1)^2\sigma^2}{24(L-1)^4(L-2)(L-3)}$$

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$ Gaussian $\quad$ + Fick's law

$$\boxed{\frac{\langle Q^{2n}\rangle_c}{t} \sim \frac{\sigma^n}{L^2}}$$ for $n \geq 2$

# UNIVERSAL CUMULANTS OF THE CURRENT

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

# UNIVERSAL CUMULANTS OF THE CURRENT

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$ Gaussian

$$\boxed{\frac{\langle Q^4\rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6\rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8\rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}}$$ Universal

$$\boxed{\langle e^{\lambda Q_t}\rangle \sim e^{t\,\mu(\lambda)}} \quad \text{with} \quad \boxed{\mu(\lambda) - \frac{\lambda^2}{2}\frac{\langle Q^2\rangle_c}{t} = \frac{1}{L^2}\,\mathcal{F}\left(-\frac{\sigma\lambda^2}{4}\right)}$$

$$\mathcal{F}(u) = -4\sum_{n\geq 1}\left[n\pi\sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u\right] = \frac{1}{3}u^2 + \frac{1}{45}u^3 + \frac{1}{378}u^4 + \ldots$$

$\boxed{\mathcal{F}\text{ universal}}$ Singularity as $u \to \frac{\pi^2}{2}$

## BETHE ANSATZ

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

The evolution

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')P(\mathcal{C}') - M(\mathcal{C}',\mathcal{C})\mathcal{P}(\mathcal{C})$$

One can decompose

$$M(\mathcal{C},\mathcal{C}') = M_1(\mathcal{C},\mathcal{C}') + M_0(\mathcal{C},\mathcal{C}') + M_{-1}(\mathcal{C},\mathcal{C}')$$

$M_q(\mathcal{C},\mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} M_1 + M_0 + e^{-\lambda} M_{-1}}$$

# BETHE ANSATZ EQUATIONS

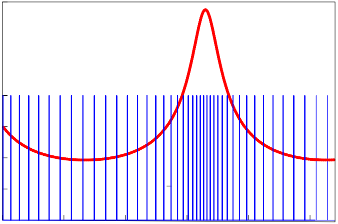
Define $s = \lambda/L$

$$z_i^L = \prod_{\substack{j=1 \\ j\neq i}}^{N} \left[ -\frac{e^s - 2z_i + e^{-s} z_i z_j}{e^s - 2z_j + e^{-s} z_i z_j} \right],$$

Then

$$\mu(\lambda) = \sum_i^N z_i e^{-s} + \frac{1}{z_i} e^s - 2$$

- The $z_i$ accumulate on a line
- Universality comes from the discrete nature of the $z_i$



# FLUCTUATING HYDRODYNAMICS

Gaussian expansion of the **macroscopic fluctuation theory** around a constant current and a flat profile.

$$\rho(x,t) = \rho + \sum_{k,\omega} k[a_{k,\omega} e^{i\omega\tau + ikx} + a^*_{k,\omega} e^{-i\omega t - ikx}]$$

$$j = j_0 - \omega[a_{k,\omega} e^{i\omega\tau + ikx} + a^*_{k,\omega} e^{-i\omega t - ikx}] \ .$$

Gaussian fluctuations

$$\mathrm{Pro}(Q_t = j_0 t, \{a_{k,\omega}\}) \sim$$

$$\exp\left[ -\frac{j_0^2}{2\sigma}\frac{t}{L} - \frac{t}{L}\sum_{\omega,k} |a_{k,\omega}|^2 \left( \frac{(\sigma\omega + j_0\sigma' k)^2}{\sigma^3} + \frac{D^2 k^4}{\sigma} - \frac{j_0^2 \sigma'' k^2}{2\sigma^2} \right) \right]$$

- Integrate over the fluctuations
- Sum over the discrete modes $k$

# RESULTS FOR A GENERAL DIFFUSIVE SYSTEM

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}}$$

$$\mu(\lambda) - \frac{\lambda^2 \langle Q^2 \rangle}{2t} = \frac{1}{L^2} D \mathcal{F}\left( \frac{\sigma \sigma''}{16 D^2} \lambda^2 \right)$$

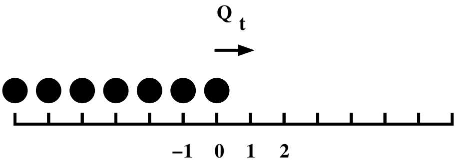
$$\mathcal{F}(u) = -4 \sum_{n \geq 1} \left[ n\pi \sqrt{n^2 \pi^2 - 2u} - n^2 \pi^2 + u \right] = \frac{1}{3} u^2 + \frac{1}{45} u^3 + \frac{1}{378} u^4 + \ldots$$

- Phase transition as $u \to \pi^2/2$
- For $n \geq 2$

$$\boxed{\frac{\langle Q^{2n} \rangle_c}{t} \sim \frac{1}{L^2} \;\; \frac{(2n)! B_{2n-2}}{n! \; (n-1)!} \;\; D \left( \frac{\sigma \sigma''}{8 D^2} \right)^n}$$

$B_n$ Bernoulli numbers

## STEP INITIAL CONDITION




# ASEP

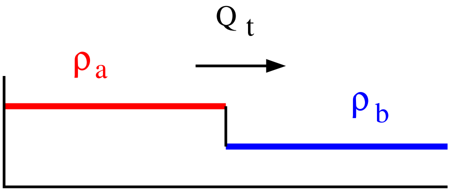
G. Schütz 98

⋮

Prähofer Spohn 2000-2002

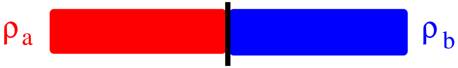
⋮

Tracy Widom 2008



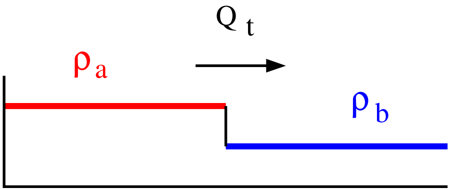

# SSSEP

D Gerschenfeld 2009




$$\left\langle e^{\lambda Q_t} \right\rangle = ?$$

# STEP INITIAL CONDITION




SSSEP

D Gerschenfeld 2009

$$\boxed{\langle e^{\lambda Q_t} \rangle \simeq \exp[\sqrt{t}\ F(\omega)]}$$

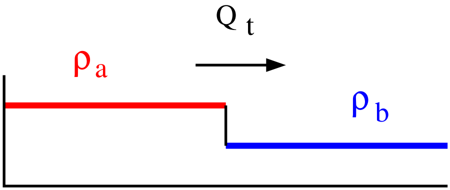
where

$$\boxed{\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a\rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)}$$

and

$$F(\omega) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}}\ \omega^n \equiv \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[1 + \omega e^{-k^2}\right]$$

# STEP INITIAL CONDITION




SSSEP

D Gerschenfeld 2009

$$\left\langle e^{\lambda Q_t} \right\rangle \simeq \exp[\sqrt{t}\ F(\omega)]$$

with

$$F(\omega) = \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[ 1 + \omega e^{-k^2} \right]$$

For large $Q_t$

$$\mathrm{Pro}(Q_t) \sim \exp[-\frac{\pi^2}{12}\ Q_t^3 / t]$$

## DIFFUSIVE SYSTEMS

Open system

$$\langle Q^n \rangle_c \sim L^{-1}$$

Ring

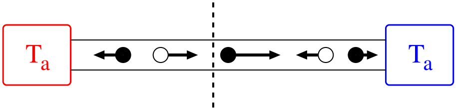
$$\langle Q^2 \rangle_c \sim L^{-1}$$

$$\langle Q^{2n} \rangle_c \sim L^{-2} \quad \text{for} \quad n \geq 2$$

Infinite system

$$\langle Q^n \rangle_c \sim t^{\frac{1}{2}}$$

# HARD PARTICLE GAS: the second cumulant



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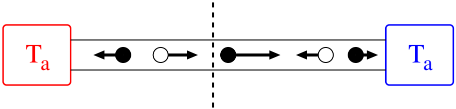
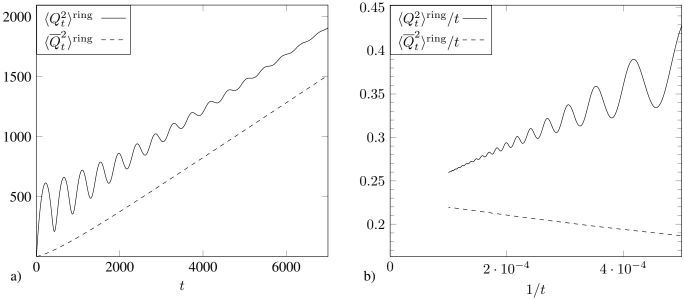
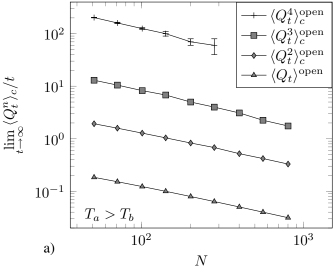


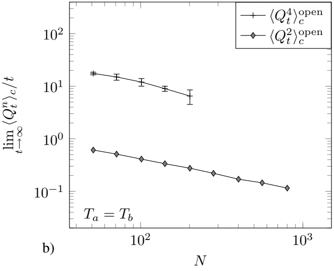




$\frac{\langle Q^2 \rangle}{t}$ has a limit

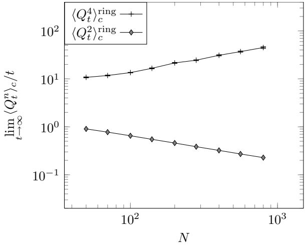
# HARD PARTICLE GAS: Size dependence of the cumulants

## OPEN SYSTEM



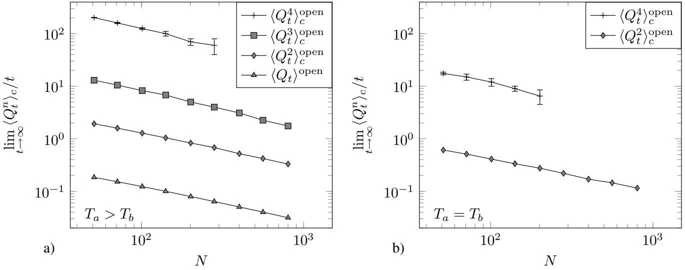




## RING

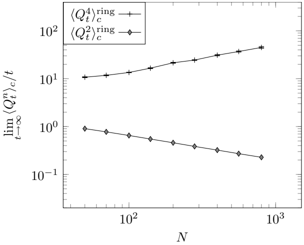


# HARD PARTICLE GAS: Size dependence of the cumulants

## OPEN SYSTEM

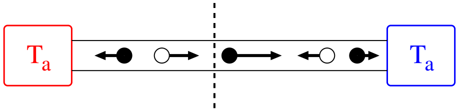



## RING

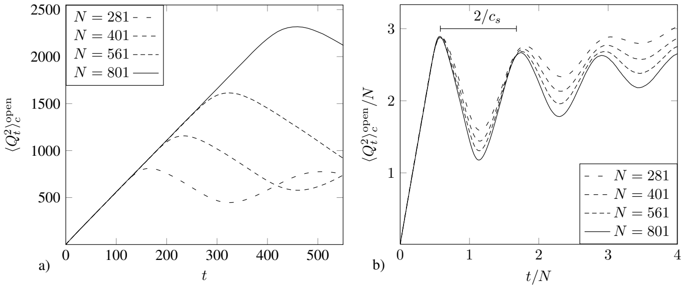



ANOMALOUS FOURIER'S LAW

# HARD PARTICLE GAS



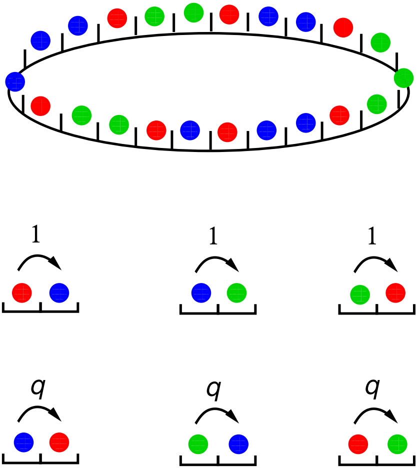

# INFINITE SYSTEM




$$\langle Q^{2n} \rangle_c \sim t$$

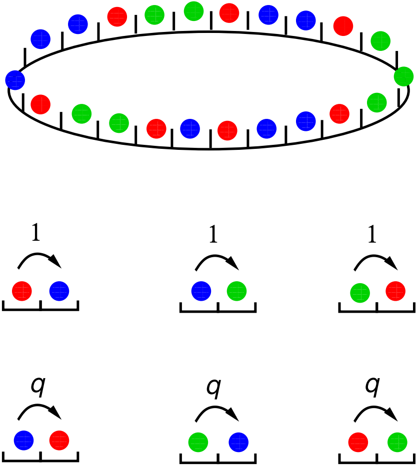
# ABC-model - PHASE TRANSITION

Evans Kafri Koduvely Mukamel 1998



# ABC-model - PHASE TRANSITION

Evans Kafri Koduvely Mukamel 1998

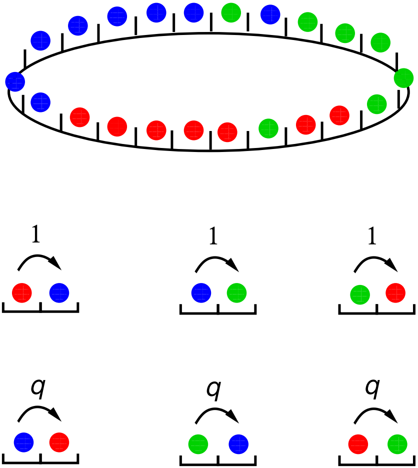



$$q = \exp\left[-\frac{\beta}{L}\right]$$

$$\beta < \beta_c = 2\pi\sqrt{3}$$

# ABC-model - PHASE TRANSITION

Evans Kafri Koduvely Mukamel 1998

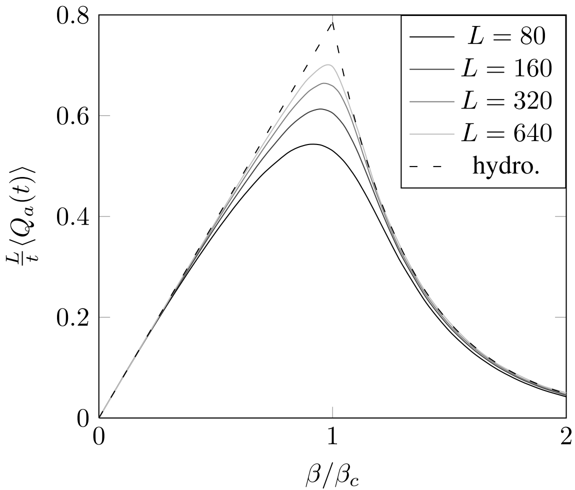



$$q = \exp\left[-\frac{\beta}{L}\right]$$

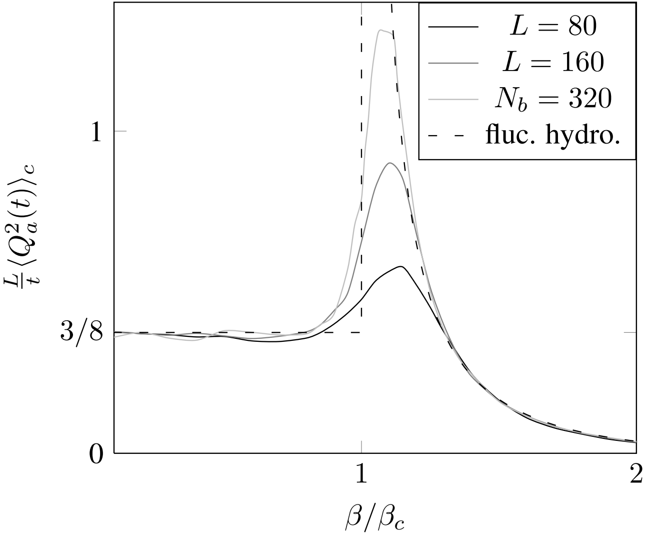
$$\beta > \beta_c = 2\pi\sqrt{3}$$

# ABC-model - Average current $\rho_A = 1/4, \rho_B = 1/2, \rho_C = 1/4)$

Gerschenfeld D 2011



# ABC-model - Current fluctuations $\rho_A = 1/4, \rho_B = 1/2, \rho_C = 1/4)$



## CONCLUSION

Universal fluctuations of the current for diffusive systems on a ring

Can be understood by the Bethe ansatz

Can be understood by the fluctuating hydrodynamics

Anomalous Fourier's law at a phase transition

## FUTURE

Theory for mechanical systems which conserved impulsion

Macroscopic fluctuation theory for a non-equilibrium initial condition